Short Papers

Advances in Modal Logic
AiML 2020

Preface

The thirteenth conference in the AiML series was organized at the University of Helsinki, Finland. Due to the exceptional situation caused by Covid 19, AiML 2020 was held online on August 24–28, 2020.

This booklet contains the short papers selected for presentation at AiML 2020. There were 29 submissions for short presentations at the conference, 23 of them were accepted for presentation. We thank all authors for their contributions.

July 20th, 2020

Nicola Olivetti
Rineke Verbrugge
Program Chairs of AiML 2020

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Contributed Papers

The proof theory of apodictic syllogistic

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1 Introduction

This research aims at extending von Plato's 2009/16 work by offering a (possibly comprehensive) account of Aristotle's deductive logic. [2] The method, which led von Plato to a transparent reconstruction of assertoric syllogistic and to the remarkable normal form theorem, is here applied to purely and mixed apodictic logic. Following it, the original source is directly analyzed, without adding anything to it. Indeed, although the success of syllogistic was such that many of its original features were changed throughout its evolution, in *An.Pr.* I, Aristotle had explicitly defined his logic and systematically introduced a *completion* proof for each imperfect mood. Our study simply consists in treating syllogistic as a ND system and derivability proofs as tree-form derivations, with great benefits both for text exegesis and for proof comprehension. Thanks to this 'translation' of the source into tree form, not only plain assertoric, but also controversial modal derivability proofs become perfectly intelligible. This proof-theoretical approach (coherent with *Analytica*'s original goal) allows us to define a clear and suited-to-study rule system and to show that all Aristotle's proofs are correct. Indeed, when considered as a deductive system, apodictic syllogistic does not appear as a "realm of darkness" [12, p. 1] anymore.

2 The assertoric system **SYL**

Before presenting his syllogistic, Aristotle introduces the language (*An.Pr.* 24a16-20). Assertoric (atomic) propositions express the belonging of a term, the predicate, to another term, the subject. They are characterized by *quality*, affirmative or negative, and *quantity*, universal or particular: [3]

$$\Pi^+(S,P) \mid \Pi^-(S,P) \mid \Sigma^+(S,P) \mid \Sigma^-(S,P)$$

[1] I wish to thank G. Corsi, E. Orlandelli and J. von Plato for guiding me in the present study and supporting it through the ERC Advanced Grant GODELIANA, led by him.

[2] The idea of extending [16] to modal syllogistic was suggested to me by Jan von Plato.

[3] We have used the compact and suggestive notation of [16]. Capital letters denote *terms*, Π and Σ indicate the predication quantity, the index the quality. In natural language:

Every S *is* P | *No* S *is* P | *Some* S *is* P | *Some* S *is not* P

For Aristotle, a syllogism is a two-premisses, valid *inference* defined by its *pair* of productive premisses. [4] Syllogisms are divided into three *figures*, based on the relation between the middle term and the extremes, and may be either perfect/complete or imperfect/incomplete (24b23-7). [5] The core of *An.Pr.* I concerns the 'reduction' of *all* imperfect moods to the perfect ones. By systematically inspecting possible premisses combination in each figure, Aristotle proves that either a given conclusion follows from them or that no one can. In the latter case, the premisses are said not to "syllogize" (and a counter-example is offered).

Following a tradition starting in the 1970s, we will treat assertoric syllogistic as a ND system. [6] The innovative tree-form treatment comes from [16], by which our system is inspired. **SYL** is obtained by simply 'translating' the rules, linearly presented in the original source, into tree form: [7]

$$\frac{\Sigma^{+}(S,P)^{\perp}}{\Pi^{-}(S,P)}\,\Sigma^{+}\perp \qquad \frac{\Sigma^{-}(S,P)^{\perp}}{\Pi^{+}(S,P)}\,\Sigma^{-}\perp \qquad \frac{\Pi^{-}(S,P)}{\Sigma^{+}(S,P)^{\perp}}\,\Pi^{-}\perp \qquad \frac{\Pi^{+}(S,P)}{\Sigma^{-}(S,P)^{\perp}}\,\Pi^{+}\perp$$

$$\frac{\Sigma^{+}(S,P)}{\Sigma^{+}(P,S)}\,\Sigma^{+}C \qquad \frac{\Pi^{-}(S,P)}{\Pi^{-}(P,S)}\,\Pi^{-}C \qquad \frac{\Pi^{+}(S,P)}{\Sigma^{+}(P,S)}\,\Pi^{+}C$$

$$\frac{\Pi^{+}(B,A) \qquad \Pi^{+}(C,B)}{\Pi^{+}(C,A)}\,\textsc{Barbara} \qquad \frac{\Pi^{-}(B,A) \qquad \Pi^{+}(C,B)}{\Pi^{-}(C,A)}\,\textsc{Celarent}$$

$$\frac{\Pi^{+}(B,A) \qquad \Sigma^{+}(C,B)}{\Sigma^{+}(C,A)}\,\textsc{Darii} \qquad \frac{\Pi^{-}(B,A) \qquad \Sigma^{+}(C,B)}{\Sigma^{-}(C,A)}\,\textsc{Ferio}$$

$$\frac{P \qquad P^{\perp}}{\perp}\,\perp I \qquad \frac{\begin{matrix}[P^{\perp}]^{1}\\ \vdots\\ \perp\end{matrix}}{P}\,\textsc{Raa}, 1$$

In *An.Pr.* 1-6, Aristotle proves the derivability in **SYL** of four second-figure syllogisms and of six third-figure ones. [8]

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[4] As is well known, the meaning of the word "syllogism" is ambiguous, referring both to valid inferences in general (so, including three-premisses, relational, or hypothetical syllogisms) and to the specific *An.Pr.* system, on which we will focus here. See at least [2, pp. 23ff.] and [14, pp. 30ff.]. Furthermore, a *stricto sensu* syllogism is defined by its *pair* of productive premisses, and not by its premisses plus its conclusion, see [11].

[5] For Aristotle, first-figure syllogisms are perfect, second- and third-figure ones are not.

[6] In 1973, Corcoran, in [3], and Smiley, in [13], (independently) present a reconstruction of assertoric syllogistic in ND form, inspiring other subsequent proposals, as [14], [7], and [15]. The first tree-form reconstructions appear in [16] and [4] (actually, an early tree-form perfection proof for Camestres can be found in [5, p. 76]).

[7] $\Sigma^{+/-}$-$\Pi^{+/-}\perp$ are presented in *De Int.* 17a33-b19 (they are actually eight but only the given four are used in *perfection* proofs), $\Pi^{-}C$ in *An.Pr.* 25a6-7, Π^{+} C in 25a7-8, $\Sigma^{+}C$ in 25a9-10, first-figure moods in *An.Pr.* b37-40, 26a1-2, 26a23-5, 26a25-6, $\perp I$ in *Metaph.* 1005b19-20, 1005b25-20, 1011b13-4, Raa in *An.Pr.* 41a23-32.

[8] Respectively, Cesare (27a5-8), Festino (27a32-7), Baroco (27a37-b2), Camestres (27b2-4), and Darapti (28a22-6), Felapton (28a26-30), Disamis (28b8-11), Datisi (28b11-3), Bocardo (28b17-20), Ferison (28b31-6). For space reasons, we had to omit their reconstructed derivability proofs, which can be found in [1] or [16]. The reconstruction shows

3 The apodictic systems $\mathbf{AP_1}$ and AP

Aristotle's modal syllogistic is universally considered as controversial and a variety of attempts for a consistent interpretation have appeared in the literature.[9] However, most of them focus on semantics, whereas we aim at reconstructing deductive systems and proofs as they are *effectively presented* in the source, without (at this stage) suggesting an interpretation. As for the assertoric fragment in [16], the apodictic system is directly obtained from the original text and perfecting proofs are straightforwardly reconstructed. This transparently shows them to be well-constructed, as Aristotle always applies his rules coherently.

Syllogistic propositions are characterized not only by quality and quantity, but also by *modality*: *assertoric, necessary* and *possible* (25a1-2). The apodictic language is obtained by endowing the assertoric one with apodictic predication: S is *necessarily* P.[10]

$$\Pi^+[S,P] \mid \Pi^-[S,P] \mid \Sigma^+[S,P] \mid \Sigma^-[S,P]$$

Aristotle takes into account all the five possible combinations of modal and assertoric premisses. We will here analyze the purely- and mixed-apodictic fragments only. The treatment of purely apodictic logic is extremely concise. Apodictic conversions (25a26-36) and perfect syllogisms (29b35-30a3) are defined analogously to the assertoric ones. Purely-apodictic $\mathbf{AP_1}$ is as follows:

$$\frac{\Pi^-[\mathrm{S,P}]}{\Pi^-[\mathrm{P,S}]}\ \Pi^-[\mathrm{C}] \qquad \frac{\Pi^+[\mathrm{S,P}]}{\Sigma^+[\mathrm{P,S}]}\ \Pi^+[\mathrm{C}] \qquad \frac{\Sigma^+[\mathrm{S,P}]}{\Sigma^+[\mathrm{P,S}]}\ \Sigma^+[\mathrm{C}]$$

$$\frac{\Pi^+[\mathrm{B,A}] \qquad \Pi^+[\mathrm{C,B}]}{\Pi^+[\mathrm{C,A}]}\ \mathrm{B[A]RB[A]R[A]} \qquad \frac{\Pi^-[\mathrm{B,A}] \qquad \Pi^+[\mathrm{C,B}]}{\Pi^-[\mathrm{C,A}]}\ \mathrm{C[E]L[A]R[E]NT}$$

$$\frac{\Pi^+[\mathrm{B,A}] \qquad \Sigma^+[\mathrm{C,B}]}{\Sigma^+[\mathrm{C,A}]}\ \mathrm{D[A]R[I][I]} \qquad \frac{\Pi^-[\mathrm{B,A}] \qquad \Sigma^+[\mathrm{C,B}]}{\Sigma^-[\mathrm{C,A}]}\ \mathrm{F[E]R[I][O]}$$

$$\frac{\Pi^+[\mathrm{A,B}] \qquad \Sigma^-[\mathrm{C,B}]}{\Sigma^-[\mathrm{C,A}]}\ \mathrm{B[A]R[O]C[O]} \qquad \frac{\Sigma^-[\mathrm{B,A}] \qquad \Pi^+[\mathrm{B,C}]}{\Sigma^-[\mathrm{C,A}]}\ \mathrm{B[O]C[A]RD[O]}$$

Differently from **SYL**, in $\mathbf{AP_1}$ B[A]R[O]C[O] and B[O]C[A]RD[O] are primi-

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that Aristotle actually introduces two distinct, but equivalent (29b6-15), systems. The more economical one does not include Σ^+C, Darii and Ferio. Furthermore, no fourth figure exists and the 14 valid syllogisms do not include the, subsequently introduced, subaltern ones, as their premisses do not differ from those of the corresponding superaltern moods.

[9] It is *communis opinio* that Aristotle's modal syllogistic raises several problems. Nevertheless, many scholars have introduced formal model(s) offering reconstructions, usually partial (one exception is [6]). The literature on the topic is vast – for an updated *status quaestionis*, see [12, pp. 32-37] – but most of the works are focussed on giving a semantics for modal syllogistic (sometimes departing from text evidence). We avoid this and only present Aristotle's words in a today more 'digestible' form. To the best of our knowledge, there is no work presenting Aristotle's modal syllogistic as a ND-system and its completion proofs as tree-form derivations. The most resembling study seems to be McCall's axiomatization (likewise showing Aristotle's consistent use of his inference rules, [8, p. 95]).

[10] Modalities are not logical operators but part of the structure of the atomic formulas to which the four quantifiers are applied.

tive.[11] $\mathbf{AP_1}$ derivability proofs are obtained as the corresponding assertoric (ostensive) ones.[12] There are four second- and six third-figure moods.

Mixed-apodictic syllogisms are such that one premiss is assertoric, the other apodictic and from them something apodictic is concluded. **AP** is obtained by adding to the rules of **SYL** and $\mathbf{AP_1}$ first-figure mixed syllogisms (30a18-b2):

$$\dfrac{\Pi^{+}[B,A] \qquad \Pi^{+}(C,B)}{\Pi^{+}[C,A]}\ \text{B[A]RBAR[A]} \qquad \dfrac{\Pi^{-}[B,A] \qquad \Pi^{+}(C,B)}{\Pi^{-}[C,A]}\ \text{C[E]LAR[E]NT}$$

$$\dfrac{\Pi^{+}[B,A] \qquad \Sigma^{+}(C,B)}{\Sigma^{+}[C,A]}\ \text{D[A]RI[I]} \qquad \dfrac{\Pi^{-}[B,A] \qquad \Sigma^{+}(C,B)}{\Sigma^{-}[C,A]}\ \text{F[E]R[I]O}$$

Aristotle shows that there are nine imperfect mixed-apodictic syllogisms: three in the second figure and six in the third. Second-figure moods are C[E]SAR[E] (30b6-14), CAM[E]STR[E]S (30b14-9) and F[E]STIN[O] (31a1-11), which are (respectively) shown derivable as follows:[13]

$$\dfrac{\dfrac{\Pi^{-}[B,A]}{\Pi^{-}[A,B]}\ \Pi^{-}[C] \qquad \Pi^{+}(C,A)}{\Pi^{-}[C,B]}\ \text{C[E]LAR[E]NT}$$

For first let the privative be necessary and let it not be possible for A to belong to any B, but let A merely belong to C. Then, since the privative converts, neither is it possible for B to belong to any A. But A belongs to every C; consequently, it is not possible for B to belong to any C, for C is below A. [*An.Pr.* 30b6-14]

$$\dfrac{\dfrac{\Pi^{-}[C,A]}{\Pi^{-}[A,C]}\ \Pi^{-}[C] \qquad \Pi^{+}(B,A)}{\dfrac{\Pi^{-}[B,C]}{\Pi^{-}[C,B]}\ \Pi^{-}[C]}\ \text{C[E]LAR[E]NT} \qquad \dfrac{\dfrac{\Pi^{-}[B,A]}{\Pi^{-}[A,B]}\ \Pi^{-}[C] \qquad \Sigma^{+}(C,A)}{\Sigma^{-}[C,B]}\ \text{F[E]RI[O]}$$

There is no mixed syllogism corresponding to BAROCO (31a11-16). Third-figure D[A]RAPT[I] (31a24-31), DAR[A]PT[I] (31a31-5), F[E]LAPT[O]N (31a35-8), D[A]TIS[I] (31b12-7), DIS[A]M[I]S (31b17-20) and F[E]RIS[O]N (31b33-6) are shown derivable respectively as:

$$\dfrac{\Pi^{+}[C,A] \qquad \dfrac{\Pi^{+}(C,B)}{\Sigma^{+}(B,C)}\ \Pi^{+}C}{\Sigma^{+}[B,A]}\ \text{D[A]RI[I]} \qquad \dfrac{\Pi^{+}[C,B] \qquad \dfrac{\Pi^{+}(C,A)}{\Sigma^{+}(A,C)}\ \Pi^{+}C}{\dfrac{\Sigma^{+}[A,B]}{\Sigma^{+}[B,A]}\ \Sigma^{+}[C]}\ \text{D[A]RI[I]}$$

$$\dfrac{\Pi^{-}[C,A] \qquad \dfrac{\Pi^{+}(C,B)}{\Sigma^{+}(B,C)}\ \Pi^{+}C}{\Sigma^{-}[B,A]}\ \text{F[E]RI[O]} \qquad \dfrac{\Pi^{+}[C,B] \qquad \dfrac{\Sigma^{+}(C,A)}{\Sigma^{+}(A,C)}\ \Sigma^{+}C}{\Sigma^{+}[A,B]}\ \text{D[A]RI[I]}$$

[11] Actually, the term variables of B[A]R[O]C[O] and B[O]C[A]RD[O] are respectively that of second- (N-M-X) and third-figure moods (S-P-R).

[12] For the derivations, see [1, pp. 83-88].

[13] For space reasons, we compare our reconstruction with *An.Pr.* text for C[E]SAR[E]'s proof only, but for each derivation the corresponding source reference is quoted to make easily possible to check that these (correct) derivability proofs are genuinely Aristotelian.

$$\dfrac{\dfrac{\Pi^+[\mathrm{C,B}] \qquad \dfrac{\Sigma^+(\mathrm{C,A})}{\Sigma^+(\mathrm{A,C})}\,\Sigma^+\mathrm{C}}{\Sigma^+[\mathrm{A,B}]}\,\mathrm{D[A]RI[I]}}{\Sigma^+[\mathrm{B,A}]}\,\Sigma^+[\mathrm{C}] \qquad\qquad \dfrac{\Pi^-[\mathrm{C,A}] \qquad \dfrac{\Sigma^+(\mathrm{C,B})}{\Sigma^+(\mathrm{B,C})}\,\Sigma^+\mathrm{C}}{\Sigma^-[\mathrm{B,A}]}\,\mathrm{F[E]RI[O]}$$

To conclude, Aristotle proves the derivability in **AP** of 14 assertoric, 14 purely-apodictic, and 13 mixed-apodictic moods and each of his perfection proofs is correct. In the future, we aim at extending this analysis to the whole Aristotelian syllogistic and at presenting a unique and suited-to-study system for all modalities. Furthermore, this proof system(s) may both be used as a tool to help reconstruct Aristotle's semantics and be developed, independently from its historical origin, in the context of Natural Logic. [14]

References

[1] Antonelli, M., *The syllogistic revival (ms)* (2019).
[2] Barnes, J., *Proof and the syllogism*, in: *Aristotle on Science. The Posterior Analytics* (1981), pp. 161–181.
[3] Corcoran, J., *A mathematical model of Aristotle's syllogistic*, Archiv für Geschichte der Philosophie **55** (1973), pp. 191–219.
[4] Dyckhoff, R., *Indirect proof and inversions of syllogisms*, Bulletin of symbolic logic **25** (2019), pp. 196–207.
[5] Kneale, W. and M. Kneale, "The Development of Logic," Clarendon Press, 1962.
[6] Malink, M., *A reconstruction of Aristotle's modal syllogistic*, History and philosophy of logic **27** (2006), pp. 95–141.
[7] Martin, J., *Aristotle's natural deduction reconsidered*, History and philosophy of logic **18** (1997), pp. 1–15.
[8] McCall, S., "Aristotle's modal syllogisms," North Holland, 1963.
[9] Moss, L., *Completeness theorems for syllogistic fragments*, in: *Logics for Lingustic Structures* (2008), pp. 317–343.
[10] Moss, L., *Natural logic*, in: *The Handbook of Contemporary Semantic Theory* (2015), pp. 561–592.
[11] Read, S., *Aristotle's theory of the assertoric syllogism* (2019), https://www.st-andrews.ac.uk.
[12] Rini, A., "Aristotle's modal proofs: Prior Analytics A8-22," Springer, 2011.
[13] Smiley, T., *What is a syllogism?*, Journal of philosophical logic **2** (1973), pp. 136–154.
[14] Smith, R., *Logic*, in: *The Cambridge Companion to Aristotle* (1995), pp. 27–65.
[15] Tennant, N., *Aristotle's syllogistic and core logic*, History and philosophy of logic **35** (2013), pp. 1–28.
[16] von Plato, J., *Aristotle's deductive logic: A proof-theoretical study*, in: *Concepts of Proof in Mathematics, Philosophy, and Computer science* (2009/16), pp. 323–346.

[14] The program of Natural Logic, delineated in the 1970s/1980s by van Benthem and Sánchez Valencia, stimulates a renewed interest for 'logics in natural language', so for syllogistic. Among the many works in this area, Moss' approach is close to ours in presenting ND syllogistic systems and in starting from the study of (everyday) *inferences*. (However, Moss' assertoric system is not exactly the same as ours, as Darapti and Π^+C are not derivable in it.) For Moss' assertoric fragment, see [9], and for an overview of its extensions, see [10]. For further details on the relationships between **AP** and Natural Logic, see [1, ch. 3-4].

Belief based on inconsistent information

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Abstract

A recent line of research has developed around logics of belief based on evidence [1,4]. One approach is based on [4] and understands belief as based on information confirmed by a reliable source. We present the work introduced in [3] where we propose a finer analysis how belief can be based on information, where the confirmation comes from multiple possibly conflicting sources and is of a probabilistic nature. We use Belnap-Dunn logic and non-standard probabilities, to account for potentially contradictory information on which belief is grounded. We combine it with an extension of Łukasiewicz logic, or a bilattice logic, within a two-layer modal logical framework to account for belief.

Keywords: epistemic logics, non-standard probabilities, Belnap-Dunn logic, two-layer modal logic.

There are several proposals of logical frameworks in the literature allowing for non-trivial inconsistencies. Belnap-Dunn logic BD [2], also referred as First Degree Entailment was specically designed to deal with possibly incomplete and inconsistent information. One of the underlying ideas of this logic is that not only amount of truth, but also amount of information that each of the values carries matters. This idea was generalized by introducing the notion of bilattices [11,9], which are algebraic structures that contain two partial orders simultaneously: a truth order, and a knowledge (or an information) order.

Belnap-Dunn four-valued logic BD, in the propositional language built using connectives $\{\wedge, \vee, \neg\}$, evaluates formulas to Belnap-Dunn square – the (de Morgan) lattice $\mathbf{4}$ built over an extended set of truth values $\{t, f, b, n\}$ (Figure 1, middle). Following Dunn's approach [7], we adopt a *double valuation model* $M = \langle W, \Vdash^+, \Vdash^- \rangle$, giving the positive and negative support of formulas in the states, which can be seen as locally evaluating formulas in the product bilattice $2 \odot 2$ (Figure 1 left), and thus in $\mathbf{4}$ (Figure 1, middle). BD logic has a simple axiomatization which is known to be (strongly) complete w.r.t. double valuation frame semantics. BD is also known to be *locally finite.* [2]

[1] The research of Marta Bílková was supported by the grant GA17-04630S of the Czech Science Foundation. The research of Sabine Frittella and Sajad Nazari was funded by the grant ANR JCJC 2019, project PRELAP (ANR-19-CE48-0006). The research of Ondrej Majer was supported by the grant GA16-15621S.

[2] It means there are only finitely many (up to inter-derivability) formulas in a fixed finite set of propositional variables. It affects the completeness of the logic in Example 0.1.

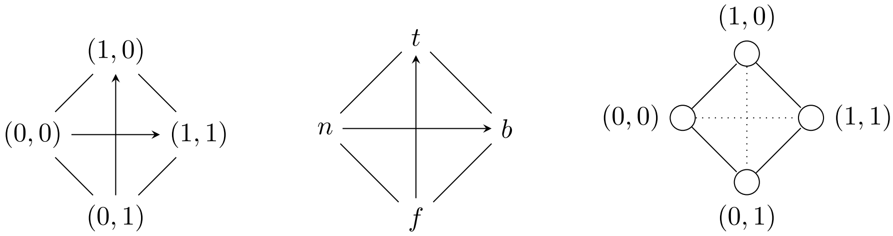

Fig. 1. The product bilattice $2 \odot 2$ (left), which is isomorphic to Dunn-Belnap square $\mathbf{4}$ (middle), and its continuous probabilistic extension (right). Negation flips the values along the horizontal line.

The idea of independence of positive and negative information naturally generalizes to probabilistic extensions of BD logic. A probabilistic Belnap-Dunn (BD) model [10] is a double valuation BD model extended with a classical probability measure on the power set of states $P(W)$ generated by a mass function on the set of states W.[3] The non-standard (positive and negative) probabilities of a formula are defined as (classical) measures of its positive and negative extensions: $p^+(\varphi) := \sum_{s \Vdash^+ \varphi} m(s)$, $p^-(\varphi) := \sum_{s \Vdash^- \varphi} m(s)$. Non-standard probabilities satisfy $0 \leq p(\varphi) \leq 1$, are monotone (resp. $p^-(\varphi)$ is antitone) w.r.t. $\vdash_{\mathsf{BD}}$, and $p(\varphi \wedge \psi) + p(\varphi \vee \psi) = p(\varphi) + p(\psi)$ [10, Lemma 1]. These axioms are weaker than classical Kolmogorovian ones and $p^+(\neg\varphi) \neq 1 - p^+(\varphi)$ in general which allows for a non-trivial treatment of inconsistent information. We can diagrammatically represent non-standard probabilities on a continuous extension of Belnap-Dunn square (Figure 1, right), which we can see as a product bilattice $\mathbf{L}_{[0,1]} \odot \mathbf{L}_{[0,1]}$. For example, the point $(0, 0)$ corresponds to no information being available, while $(1, 1)$ is the point of maximally conflicting information. The vertical dashed line corresponds to the "classical" case when positive and negative support sum up to 1.

We look at an agent who considers a set of issues represented by atomic variables, has access to sources providing information based on non-standard probabilities and builds beliefs based on these sources using some aggregation strategy. In many scenarios we can adapt aggregation strategies that have been introduced on classical probabilities: imagine for example a company that has access to a huge amount of heterogeneous data from various sources and uses software capable of analyzing these data. In this case it makes sense to consider aggregation methods that require some computational power. A natural strategy here is to evaluate sources with respect to their reliability and aggregate them by taking their weighted average. Another kind of agent is an investigator of a criminal case who builds her opinion on the guilt of a suspect based on different pieces of evidence. We first assume that all the sources are equally reliable and the investigator is very cautious and does not want to draw conclusions hastily. Hence, she relies on statements as little as all her

[3] The probability of a set $X \subseteq W$ is defined as the sum of masses of its elements.

sources agree on them. The aggregation she uses returns the minimum of the positive and the minimum of the negative probabilities provided by the sources (min-min). If on the other hand the investigator considers all the sources being perfectly reliable, she accepts every piece of evidence and builds her belief using the max-max aggregation.

To make a clear distinction between the level of information on which the agent bases her beliefs, and the level of reasoning about her beliefs, we use a *two-layer* logical framework. The formalism originated with Hájek [8], and was further developed in [5] into an abstract framework with a general theory of syntax, semantics and completeness. Syntax $(\mathcal{L}_e, \mathcal{L}_u, \mathcal{M})$ of a two layer logic $\mathcal{L}$ consists of a lower language $\mathcal{L}_e$, an upper language $\mathcal{L}_u$, and a set of modalities $\mathcal{M}$ which, applied to a non-modal formula of $\mathcal{L}_e$, form a modal atomic formula of $\mathcal{L}_u$. Semantics of a two layer logic $\mathcal{L}$ is based on *frames* of the form $F = (W, E, U, \langle \mu^{\heartsuit} \rangle_{\heartsuit \in \mathcal{M}})$, where E is a local algebra of evaluation of $\mathcal{L}_e$ in the states, U is an upper-level algebra, and for each modality its semantics is given by the map $\mu^{\heartsuit} : \prod_{s \in W} E \to U$ [4]. The resulting logic as an axiomatic system $L = (L_e, M, L_u)$ consists of an axiomatics of L_e, modal axioms and rules M, and an axiomatics of L_u.

Example 0.1 [**Logic of probabilistic belief**] In some scenarios it is reasonable to represent agents (partial) beliefs as non-standard probabilities. In this two-layer logic, the bottom layer is that of events or facts, represented by BD-information states. A source provides probabilistic information given as a mass function on the states. The modality is that of non-standard probabilistic belief, the top layer – the logic of thus formed beliefs – is based on the following extension of Łukasiewicz logic Ł [6].

We consider the product of the standard algebra of Łukasiewicz logic $[0,1]_{Ł} = ([0,1], \wedge, \vee, \&_{Ł}, \to_{Ł})$ with $[0,1]_{Ł}^{op} = ([0,1]^{op}, \vee, \wedge, \oplus_{Ł}, \ominus_{Ł})$, which arises turning the standard algebra upside down: it is an MV algebra $[0,1]_{Ł} \times [0,1]_{Ł}^{op} = ([0,1] \times [0,1]^{op}, \wedge, \vee, \&, \to)$, where $(1,0)$ is the designated value. Its logic is Łukasiewicz logic Ł. We extend the signature of the algebra with the bilattice negation $\neg(a_1, a_2) = (a_2, a_1)$, and extend the language to $\{\to, \sim, \neg\}$. We obtain the following axioms and rules, denoting the resulting consequence relation $\vdash_{Ł(\neg)}$:

$$\begin{array}{cc}
\alpha \to (\beta \to \alpha) & \neg\neg\alpha \leftrightarrow \alpha \\
(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma)) & \neg{\sim}\alpha \leftrightarrow {\sim}\neg\alpha \\
((\alpha \to \beta) \to \beta) \to ((\beta \to \alpha) \to \alpha) & ({\sim}\neg\alpha \to {\sim}\neg\beta) \leftrightarrow {\sim}\neg(\alpha \to \beta) \\
({\sim}\beta \to {\sim}\alpha) \to (\alpha \to \beta) & \alpha, \alpha \to \beta / \beta \quad \alpha / {\sim}\neg\alpha
\end{array}$$

We can provide a reduction of $\vdash_{Ł(\neg)}$ to provability in Ł and show that the extension of Ł by $\neg$ is conservative. Using finite completeness of Ł, we can prove that Ł($\neg$) is *finitely strongly complete* w.r.t. $[0,1]_{Ł} \times [0,1]_{Ł}^{op}$.

[4] For this paper, we always consider the lower algebras be all the same. But different algebras can be later used when modelling heterogeneous information. We write algebras, but often we use matrices, i.e. algebras with a set of designated values.

The two-layer syntax consists of $\mathcal{L}_e = \{\wedge, \vee, \neg\}$ language of BD, $\mathcal{M} = \{B\}$ a belief modality, and $\mathcal{L}_u = \{\rightarrow, \sim, \neg\}$ language of Ł$(\neg)$. The intended frames are $F = (W, \mathbf{4}, [0,1]_{Ł} \times [0,1]^{op}_{Ł}, \mu^B)$,[5] where μ^B is computed as follows. A source is given by a mass function on the states $m : W \rightarrow [0,1]$. Given $\mathbf{e} \in \prod_{v \in W} \mathbf{4}$, μ^B computes the following sums of weights over states: $\mu^B(\mathbf{e}) = (\sum_{\mathbf{e}_v \in \{t,b\}} m(v), \sum_{\mathbf{e}_v \in \{f,b\}} m(v))$. Thus, for a non-modal formula $\varphi \in \mathcal{L}_e$, applying μ^B to the tuple of its values in the states, we obtain the value of $B\varphi$ in $[0,1]_{Ł} \times [0,1]^{op}_{Ł}$ as a pair of its non-standard positive and negative probabilities $(\sum_{v \Vdash^+ \varphi} m(v), \sum_{v \Vdash^- \varphi} m(v)) = (p^+(\varphi), p^-(\varphi))$.[6]

The modal part M consists of axioms and a rule reflecting the axioms of non-standard probabilities:

$$B(\varphi \vee \psi) \leftrightarrow (B\varphi \ominus B(\varphi \wedge \psi)) \oplus B\psi \qquad B\neg\varphi \leftrightarrow \neg B\varphi$$
$$\varphi \vdash_{\mathsf{BD}} \psi / \vdash_{Ł(\neg)} B\varphi \rightarrow B\psi$$

The resulting logic is $(\mathsf{BD}, M, Ł(\neg))$. As BD is locally finite and strongly complete w.r.t. $\mathbf{4}$, and Ł$(\neg)$ is finitely strongly complete w.r.t. $[0,1]_{Ł} \times [0,1]^{op}_{Ł}$, we can by [5, Theorems 1 and 2] conclude that $(\mathsf{BD}, M, Ł(\neg))$ is *finitely strongly complete* w.r.t. $\mathbf{4}$ based, $[0,1]_{Ł} \times [0,1]^{op}_{Ł}$-measured frames validating M. In such frames, μ^B interprets B as a non-standard probability. From [10, Theorem 4], we know that it is the induced non-standard probability function of exactly one mass function on the BD states, which in fact yields completeness w.r.t. the intended frames described above. Since the (weighted) average aggregation of non-standard probabilities yields a non-standard probability, $(\mathsf{BD}, M, Ł(\neg))$ is also adequate to capture frames with *multiple sources* such that $\mu^B : P(\prod_{s \in W} \mathbf{4}) \rightarrow [0,1]_{Ł} \times [0,1]^{op}_{Ł}$ computes the (weighted) average of the probabilities given by the individual sources.

Alternatively, we can take $\mathcal{L}_u = \{\wedge, \vee, \sqcap, \sqcup, \subset, \neg, 0\}$ as the language of the *product residuated bilattice* $[0,1]_{Ł} \odot [0,1]_{Ł} = ([0,1] \times [0,1], \wedge, \vee, \sqcap, \sqcup, \supset, \neg, (0,0))$, defined in the spirit of [9] (considered as a matrix with $F = \{(1,a) \mid a \in [0,1]\}$ being the designated values). With a little work, we can define $\oplus, \ominus$ and use literally the same modal axioms M as above.

Example 0.2 [**Logic of monotone coherent belief**] The intended frames are $F = (W, \mathbf{4}, \mathbf{L}_{[0,1]} \odot \mathbf{L}_{[0,1]}, \mu^B)$ where $\mathbf{L}_{[0,1]} \odot \mathbf{L}_{[0,1]}$ is the bilattice on Figure 1 (right), we have multiple sources and $\mu^B : P(\prod_{s \in W} \mathbf{4}) \rightarrow \mathbf{L}_{[0,1]} \odot \mathbf{L}_{[0,1]}$ computes the min-min (max-max) aggregation of the probabilities given by the individual sources. In general this does not yield a non-standard probability, only the interdefinability of positive and negative support via negation is preserved. This motivates considering logic $(\mathsf{BD}, M, \mathsf{BD})$, where the modal part M consists of the axiom and rule

$$B\neg\varphi \dashv\vdash_{\mathsf{BD}_u} \neg B\varphi \qquad \varphi \vdash_{\mathsf{BD}_e} \psi / B\varphi \vdash_{\mathsf{BD}_u} B\psi.$$

[5] Formulas of $\mathcal{L}_e$ are evaluated locally in the states of W using $\mathbf{4}$, as in the frame semantics for BD.

[6] The value of φ in v being among $\{t, b\}$ means it is positively supported in v, i.e. $v \Vdash^+ \varphi$. Similarly for negative support.

As BD is strongly complete w.r.t. both $\mathbf{4}$ and $\mathbf{L}_{[0,1]} \odot \mathbf{L}_{[0,1]}$,[7] we obtain that $(\mathsf{BD}, M, \mathsf{BD})$ is strongly complete w.r.t. $\mathbf{4}$-based $\mathbf{L}_{[0,1]} \odot \mathbf{L}_{[0,1]}$-measured frames validating M.[8]

Further directions. A natural aggregation strategy to consider would be Dempster–Shafer combination rule [12] (which is problematic in cases of high conflict, because it can provide counter intuitive results) adapted to the BD-based setting. As a source does not often give an opinion about each formula of the language, we need to account for sources providing partial probability maps. Another quest is to capture dynamics of information and belief given by updates on the level of sources, and to generalize the framework to the multi agent setting, involving group modalities and dynamics of belief. Specifically, forming group belief, including common and distributed belief, will involve communication and/or sharing and pooling of sources. It might also call for a use of modalities inside the upper logic to account for reflected beliefs, in contrast to the beliefs grounded directly in the sources.

References

[1] Baltag, A., N. Bezhanishvili, A. Özgün and S. Smets, *Justified belief and the topology of evidence*, in: *Proceedings of the 23rd International Workshop on Logic, Language, Information, and Computation* - LNCS Volume 9803, Springer, pp. 83–103 (2016).

[2] Belnap, N., *How a computer should think*, in: H. Omori and H. Wansing, editors, *New Essays on Belnap-Dunn Logic*, Synthese Library (Studies in Epistemology, Logic, Methodology, and Philosophy of Science) **418**, Springer, 2019.

[3] Bílková, M., S. Frittella, O. Majer and S. Nazari, *How to reason with inconsistent probabilistic information?* manuscript (2020). URL `https://arxiv.org/abs/2003.12906`

[4] Bílková, M., O. Majer and M. Peliš, *Epistemic logics for sceptical agents*, Journal of Logic and Computation, Volume 26, Issue 6, pp. 1815–1841 (2016).

[5] Cintula, P. and C. Noguera, *Modal logics of uncertainty with two-layer syntax: A general completeness theorem*, in: *Logic, Language, Information, and Computation - 21st International Workshop, WoLLIC 2014, Valparaíso, Chile, September 1-4, 2014. Proceedings*, pp. 124–136.

[6] Di Nola, A. and I. Leustean, *Lukasiewicz logic and mv-algebras*, in: P. Cintula, P. Hájek and C. Noguera, editors, *Handbook of Mathematical Fuzzy Logic Vol. 2*, College Publications, 2011.

[7] Dunn, J. M., *Intuitive semantics for first-degree entailments and 'coupled trees'*, Philosophical Studies **29** (1976), pp. 149–168.

[8] Hájek, P., "Metamathematics of Fuzzy Logic," Trends in Logic **4**, Kluwer, 1998.

[9] Jansana, R. and U. Rivieccio, *Residuated bilattices*, Soft Comput. 16, pp. 493–504 (2012).

[10] Klein, D., O. Majer and S. Raffie-Rad, *Probabilities with gaps and gluts*, manuscript (2020). URL `http://arxiv.org/abs/2003.07408`

[11] Rivieccio, U., "An Algebraic Study of Bilattice-based Logics," Ph.D. thesis, University of Barcelona - Universita degli Studi di Genova, 2010.

[12] Shafer, G., "A mathematical theory of evidence," Princeton university press.

[7] Because it has $(\mathbf{4}, \{(1,0),(1,1)\})$ as a sub-matrix: the obvious embedding is a *strict* homomorphism of de Morgan matrices - it preserves and reflects the filters.

[8] This kind of generic completeness however provides counterexamples with a single 4-valued state, which are not quite the intended semantics involving multiple states and sources.

A Proof from 1988 that PDL has Interpolation?

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Abstract

Multiple arguments that Propositional Dynamic Logic has Craig Interpolation have been published, but one has been revoked and the status of the others is unclear. Here we summarise a proof attempt originally written by the first author in German in 1988. We also make available the original text and an English translation.
The proof uses a tableau system with annotations. Interpolants are defined for partitioned nodes, going from leaves to the root with appropriate definitions for each rule. To prevent infinite branches generated by the $*$ operator, additional marking rules are used. In particular, nodes are also defined as end nodes when they have a predecessor with the same set of formulas along a branch with the same marking.
We end with open questions about the proof idea and connections to more recent related work on non-wellfounded proof systems.

Keywords: Propositional Dynamic Logic, Craig Interpolation, Tableau.

1 Introduction

Propositional Dynamic Logic (PDL) from [4] is a well-known modal logic which is both expressive and well-behaved. PDL can express common programming constructs such as conditionals and loops, but also has a small model property.

A logic has Craig Interpolation (CI) iff for any validity $\phi \to \psi$ there exists a formula θ in the vocabulary that is used both in ϕ and in ψ such that $\phi \to \theta$ and $\theta \to \psi$ are valid. The formula θ is then called an *interpolant*.

For PDL the vocabulary includes atomic propositions and atomic programs. For example, $[(A \cup B)^*](p \wedge q) \to [(B; B)^*](q \vee r)$ is valid in PDL and $[B^*]q$ is an interpolant for this validity. But whether such interpolants always exist, i.e. whether PDL has CI, has been studied for more than four decades and is still unknown. The key challenge is how to systematically find interpolants for

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validities involving the star operator a^* which denotes arbitrary finite iteration of a program a. There have been at least the following three proof attempts:

- Daniel Leivant in [10] from 1981. This article presents a sequent calculus including a rule for $*$ with infinitely many premises. This rule is then replaced with a finitary rule and an intuitionistic variant of the system is defined. Interpolation is then shown in the intuitionistic system using Maeharas Method, defining interpolants for each node in a proof [12, p. 33]. Interpolants for $*$ are defined via fixed points of matrices of programs.

  In [9] it is said that it was not "possible to verify the argument" and claimed that the finitary rule is problematic. But the rule can be validated using the finite model property of PDL, as argued in [5]. Still, other parts of the argument, e.g. the translation to the intuitionistic system, seem problematic. As far as we know, the status of this argument is currently unknown [6].
- Manfred Borzechowski in [2] from 1988. The idea here is similar to [10], but using a tableau system instead of a sequent calculus. This text is also criticised in [9], but without any specific argument.
- Tomasz Kowalski in [7] from 2002. This algebraic proof was officially retracted [8] in 2004, after a flaw was pointed out by Yde Venema.

The correctness of the first two texts is still the subject of discussions. In this note we summarise the proof attempt from [2]. This diploma thesis was written under the supervision of Wolfgang Rautenberg at FU Berlin, but not published. Together with this summary we make available the original German text and an English translation at `https://malv.in/2020/borzechowski-pdl`. Page numbers refer to the German text, but are also shown in the translation.

We use the following notation: p, q, etc. are atomic propositional variables, P, Q, etc. are formulas from $P ::= p \mid \neg P \mid P \wedge Q \mid [a]P$. Moreover, A, B, etc. are atomic programs and a, b, etc. are programs from $a ::= A \mid a; a \mid a \cup a \mid a^* \mid P?$. We do not repeat the semantics for PDL here — see the original page 6 or [4].

Section 2 provides an overview of the tableau system, Section 3 describes the main idea how to define interpolants, and Section 4 lists open questions.

2 Tableaux for PDL

The system is defined below. We read rules top-down and use "... | ..." for branches. The Boolean rules and those for PDL constructs besides $*$ are standard. The *critical rule* (At) for atomic programs uses $X_A := \{P \mid [A]P \in X\}$ which corresponds to a transition to another state in a Kripke model.

To deal with the $*$ operator and to prevent infinitely long branches, the system uses the following two non-standard features and extra condition 6.

Nodes with n-formulas. The $(\neg n)$ rule is essentially a diamond rule for the $*$ operator. It also replaces $*$ by the string '(n)'. Formulas with '(n)' are n-*formulas*, in contrast to *normal* formulas. An n-formula becomes normal again by extra condition 1, which applies iff an atomic modality is reached.

Markings. Formulas can be marked with other formulas as upper indices, using the loading rule $(M+)$. Nodes with marked formulas are called *loaded*, in contrast to *free*. Markings can be removed or changed by $(M-)$, $(\neg n)$ or $(\neg ?)$.

Definition 2.1 A *PDL tableau* is a finite tree generated according to the following rules and in addition adhering to the seven extra conditions below.
The classical rules:

$$(\neg)\ \frac{X; \neg\neg P}{X; P} \qquad (\wedge)\ \frac{X; P \wedge Q}{X; P; Q} \qquad (\neg\wedge)\ \frac{X; \neg(P \wedge Q)}{X; \neg P \quad | \quad X; \neg Q}$$

The local rules:

$$(\neg\cup)\ \frac{X; \neg[a \cup b]P}{X; \neg[a]P \quad | \quad X; \neg[b]P} \qquad (\neg ?)\ \frac{X; \neg[Q?]P}{X; Q; \neg P} \qquad (\neg ;)\ \frac{X; \neg[a; b]P}{X; \neg[a][b]P}$$

$$(\cup)\ \frac{X; [a \cup b]P}{X; [a]P; [b]P} \qquad (?)\ \frac{X; [Q?]P}{x; \neg Q \quad | \quad X; P} \qquad (;)\ \frac{X; [a; b]P}{X; [a][b]P}$$

$$(\neg n)\ \frac{X; \neg[a^*]P}{X; \neg P \quad | \quad X; \neg[a][a^{(n)}]P} \qquad (n)\ \frac{X; [a^*]P}{X; P; [a][a^{(n)}]P}$$

The PDL rules:

$$(M+)\ \frac{X; \neg[a_0] \ldots [a_n]P}{X; \neg[a_0] \ldots [a_n]P^P}\ \ X \text{ free} \qquad \text{the } \textit{loading rule},$$

$$(M-)\ \frac{X; \neg[a]P^R}{X; \neg[a]P} \qquad \text{the } \textit{liberation rule},$$

$$(At)\ \frac{X; \neg[A]P^R}{X_A; \neg P^{R \setminus P}} \qquad \text{the } \textit{critical} \text{ rule}.$$

The marked rules (where $\ldots^{R \setminus P}$ indicates that R is removed iff $R = P$):

$$(\neg\cup)\ \frac{X; \neg[a \cup b]P^R}{X; \neg[a]P^R \quad | \quad X; \neg[b]P^R} \qquad (\neg ;)\ \frac{X; \neg[a; b]P^R}{X; \neg[a][b]P^R}$$

$$(\neg n)\ \frac{X; \neg[a^*]P^R}{X; \neg P^{R \setminus P} \quad | \quad X; \neg[a][a^{(n)}]P^R} \qquad (\neg ?)\ \frac{X; \neg[Q?]P^R}{X; Q; \neg P^{R \setminus P}}$$

1. Instead of a node $X; \neg[A]P$ or $X; [A]P$ with an n-formula P we always obtain the node $X; \neg[A]f(P)$ or $X; [A]f(P)$, respectively, where $f(P)$ is obtained by replacing (n) with $*$.
2. Instead of a node $X; [a^{(n)}]P$ we always obtain the node X.
3. A rule must be applied to an n-formula whenever it is possible.
4. No rule may be applied to a $\neg[a^{(n)}]$-node.
5. To a node obtained using $(M+)$ we may not apply $(M-)$.
6. If a normal node t has a predecessor s with the same formulas and the path $s \ldots t$ uses (At) and is loaded if s is loaded, then s is an end node.
7. Every loaded node that is not an end note by condition 6 has a successor.

Claim 2.2 *The system from Definition 2.1 is sound and complete for PDL.*

The full completeness proof is contained in sections 1.8 to 1.10 of the original text. The main idea is to construct a Kripke model from an open tableau.

3 Interpolation via Tableaux

We claim that the tableau system can be used to show interpolation. We first define interpolants for partitioned sets of formulas. A partitioned set X is a disjoint union of two subsets X_1, X_2. We write it as $X = X_1/X_2$.

Definition 3.1 A formula θ is an interpolant for a partitioned set X_1/X_2 iff θ is in the vocabulary of that is used in both X_1 and X_2 and the two sets $X_1 \cup \{\neg\theta\}$ and $\{\theta\} \cup X_2$ are both inconsistent.

Corollary 3.2 *A formula θ is an interpolant for a validity $\phi \to \psi$ iff θ is an interpolant for the partitioned set X_1/X_2 given by $X_1 = \{\phi\}$ and $X_2 = \{\neg\psi\}$.*

To find an interpolant for a validity $\phi \to \psi$ we start a tableau with $\phi/\neg\psi$ as its root. This tableau is built as usual from the root to the leaves, applying the rules to partitioned sets. Then we go in the opposite direction: starting at the leaves, we define an interpolant for each node. Depending on the rule which was applied, we use the interpolant(s) of the child node(s) to define a new interpolant for the parent node. In addition, the interpolant might depend on whether the active formula in a rule application is in the left or right side of the partition. As mentioned above, this is similar to Maehara's Method for sequent calculi [12, p. 33]. We discuss two rules as examples here.

Interpolating $(\neg\cup)$. Suppose we use $(\neg\cup)$ in the right set. Given two interpolants θ_a and θ_b for $X_1/X_2; \neg[a]P$ and $X_1/X_2; \neg[b]P$ respectively, we define the new interpolant $\theta := \theta_a \wedge \theta_b$ for the parent node $X_1/X_2; \neg[a \cup b]P$. Similarly, on the left side we would use $\theta := \theta_a \vee \theta_b$ for $X_1; \neg[a \cup b]P/X_2$.

Interpolating (At). Suppose we use (At) in the left set to go from a parent node $\neg[A]\phi; Y_1/Y_2$ to a child node $\neg\phi; (Y_1)_A/(Y_2)_A$. Suppose θ_A is an interpolant for the child node. Then $\neg\phi; (Y_1)_A; \neg\theta_A$ and $(Y_2)_A; \theta_A$ are inconsistent. We now want an interpolant for the parent, i.e. a θ such that $\neg[A]\phi; Y_1; \neg\theta$ and $Y_2; \theta$ are inconsistent. A solution is to set $\theta := \langle A \rangle \theta_A$, unless Y_2 is empty, in which case we are not allowed to use A, so we ignore θ_A and let $\theta := \bot$. Similarly, if (At) is applied in the right set we use $\theta := [A]\theta_A$, unless Z_1 is empty, in which case we let $\theta := \top$.

We refer to the original text for two examples. A closed tableau for the set $\{\neg[(A \cup p?)^*]q, [A^*]q\}$ is given on page 29 and an interpolant for $[(A; A)^*](p \wedge [A; (B \cup C)]0) \to [A^*](p \vee [C]q)$ is computed in Section 2.4: $[A^*](p \vee [C]0)$.

4 Open Questions

The previous two sections provide only a high-level overview of the argument. To verify it completely we will further study the following two main questions:

- How exactly are the existence lemma and completeness of the system shown? In particular, what is the role of *first free normal successor nodes*?

- How are interpolants defined for end nodes due to condition 6? The original text uses the extra tableaux $\mathcal{T}^I$ and $\mathcal{T}^J$ for this, what is their role?

If the proof can be verified, there are of course further questions:

- Can we simplify the proof to only consider test-free PDL?
- How does the system compare to recent work on infinitary and non-wellfounded systems, such as [1] for μ-calculus and [3] for PDL?
- Can interpolation be efficiently implemented into an automated prover? We have started to implement parts of the given system, similar to how the star-free fragment of [10] was implemented by [11].

To conclude, we hope that this summary will help to further scrutinise the proof and encourage the interested participant of AiML 2020 to contact us.

References

[1] Afshari, B., G. Jäger and G. E. Leigh, *An infinitary treatment of full mu-calculus*, in: R. Iemhoff, M. Moortgat and R. de Queiroz, editors, *Workshop on Logic, Language, Information, and Computation (WoLLIC)*, 2019, pp. 17–34.
URL `https://doi.org/10.1007/978-3-662-59533-6_2`

[2] Borzechowski, M., *Tableau-Kalkül für PDL und Interpolation* (1988), Diplomarbeit.
URL `https://malv.in/2020/borzechowski-pdl/`

[3] Docherty, S. and R. N. S. Rowe, *A non-wellfounded, labelled proof system for propositional dynamic logic*, in: S. Cerrito and A. Popescu, editors, *Automated Reasoning with Analytic Tableaux and Related Methods (TABLEAUX 2019)*, Lecture Notes in Computer Science **11714**, 2019, pp. 335–352.
URL `http://arxiv.org/abs/1905.06143`

[4] Fischer, M. J. and R. E. Ladner, *Propositional dynamic logic of regular programs*, Journal of Computer and System Sciences **18** (1979), pp. 194–211.
URL `https://doi.org/10.1016/0022-0000(79)90046-1`

[5] Gattinger, M., *Craig Interpolation of PDL – A report on the proof by Daniel Leivant (1981)* (2014).
URL `https://w4eg.de/malvin/illc/pdl.pdf`

[6] Gattinger, M. and Y. Venema, *Interpolation for PDL: an open problem?*, in: *Circularity in Syntax and Semantics*, 2019, p. 26, talk and abstract only.
URL `http://www.cse.chalmers.se/~bahafs/CiSS2019/CiSS2019BoA.pdf`

[7] Kowalski, T., *PDL has interpolation*, Journal of Symbolic Logic **67** (2002), pp. 933–946.
URL `https://doi.org/10.2178/jsl/1190150141`

[8] Kowalski, T., *Retraction note for "PDL has interpolation"*, Journal of Symbolic Logic **69** (2004), pp. 935–936.
URL `https://doi.org/10.2178/jsl/1096901777`

[9] Kracht, M., "Tools and Techniques in Modal Logic," 1999.
URL `https://wwwhomes.uni-bielefeld.de/mkracht/html/tools/book.pdf`

[10] Leivant, D., *Proof theoretic methodology for propositional dynamic logic*, in: J. Díaz and I. Ramos, editors, *Formalization of Programming Concepts (ICFPC 1981)*, Lecture Notes in Computer Science **107**, 1981, p. 356–373.
URL `https://doi.org/10.1007/3-540-10699-5_111`

[11] Perin, F., *Implementing Maehara's method for star-free Propositional Dynamic Logic* (2019), Bachelor's Thesis, University of Groningen.
URL `https://fse.studenttheses.ub.rug.nl/20770/`

[12] Takeuti, G., "Proof Theory," Studies in logic and the foundations of mathematics **81**, North-Holland, 1975.

FOIL with constant domain revisited

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Abstract

This paper answers a problem left open in Fitting's [2] by showing that the quantifier-free calculus FOIL extended with axiom B: $A \to \Box\Diamond A$ is characterized by symmetric models with constant domains. The problem in brief: how can we have constant domains without the Barcan Formula?
First, it is shown that, thanks to axiom B, an inductive set of rules $CD(k)$, for $k \in \mathbb{N}$, is derivable. Then, it is shown that this set of rules enables a constant domain Lindenbaum-Henkin construction, thus proving the completeness of FOIL.B.

Keywords: FOIL, symmetry, axiomatic system, completeness, canonical model.

1 Introduction

FOIL is a family of two-sorted first-order modal logics containing both object and intensional variables where the abstraction operator λ is used to talk about the object (if any) denoted by an intension in a given world.

In [2] an axiomatization of FOIL based on the quantifier-free language is introduced by M. Fitting and "is shown to be complete for standard logics without a symmetry condition" [2, p. 1]. "It would be interesting to know if a complete axiomatisation of FOIL can be given [...] using [...] propositional modal logics involving a symmetric accessibility relation."[2, p. 21].

We show, Lemma 3.3, that, thanks to axiom B, an inductive set of rules $CD(k)$, for $k \in \mathbb{N}$ is derivable and that these rules allow a constant domain Lindenbaum-Henkin construction. The completeness of FOIL.B follows.

As to the semantics, we slightly generalize Fitting's semantics by adding a set of labels, one for each intension, to the effect that any two intensions are different even if they map the same worlds to the same objects. This has no effect on truth and solves the problem noted in [3], see Remark 4.8.

2 Syntax and Semantics of FOIL

Syntax. We consider a signature containing, for each $n, m \in \mathbb{N}$, a countable set of $n+m$-ary relational symbols, denoted by $P^{n,m}, R^{n,m} \ldots$ The lan-

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guage contains a denumerable set of *object variables* OBJ $(x, y, z,)$ and one of *intensional variables* INT (f, g, h). The logical symbols are $\bot, \to\ \Box, \lambda, =$. $\mathcal{L}$-formulas are generated by:

$$A \ ::= \ P^{n,m}(x_1, \ldots, x_n, f_1, \ldots, f_m) \,|\, x = y \,|\, \bot \,|\, A \to A \,|\, \Box A \,|\, \langle \lambda x.A \rangle f \qquad (\mathcal{L})$$

The symbols $\top, \neg, \wedge, \vee, \leftrightarrow, \Diamond$ and $\neq$ are defined as usual, and the formula:

$$\mathsf{D}f \text{ abbreviates } \langle \lambda x.\top \rangle f \text{ and expresses `}f\text{ designates'}. \qquad (\text{Def. } \mathsf{D})$$

By $A[y/x]$ we denote the formula that is obtained by substituting each free occurrence of x in A with an occurrence of y, provided that y is free for x in A. The formula $A[g/f]$ is defined analogously.

Semantics. A *model* is a tuple $\mathcal{M} = \langle W, R, D_O, D_L, D_I, V \rangle$ where:
(i) $\langle W, R \rangle$ is a symmetric frame;
(ii) D_O is a non-empty set of objects;
(iii) D_L is a non-empty set of labels $\ell_{\widehat{f}}, \ell_{\widehat{g}} \ldots$;
(iv) D_I is a set of *intensions* such that, for each $\ell_{\widehat{f}} \in D_L$, D_I contains a partial functions $\widehat{f} : W \longrightarrow D_O \times D_L$; where, if $\widehat{f}$ is defined for $w \in W$, then $\widehat{f}(w) = \langle o, \ell_{\widehat{f}} \rangle$, for some $o \in D_O$, if $\widehat{f}$ is not defined for $w \in W$, then $\widehat{f}(w) = \ell_{\widehat{f}}$;
(v) V is a *valuation function* such that $V(P^{n,m}, w) \subseteq (D_O)^n \times (D_I)^m$ and $V(=, w) = \{\langle o, o \rangle : o \in D_O\}$.

An *assignment* is a function σ mapping individual variables to member of D_O and each intensional variables to members of D_I. $\sigma^{x \rhd o}$ $(\sigma^{f \rhd i})$ behave like σ except for x (f) that is mapped to $o \in D_w$ $(\widehat{f} \in D_I$, respectively).

Satisfaction of a formula A in a world w of a model $\mathcal{M}$ under an assignment σ, to be denoted by $\sigma \models^{\mathcal{M}}_w A$ $(\sigma \models_w A$, for short), is defined standardly for atoms and for $x = y, \bot, B \to C, \Box B$, and it is thus defined for $\langle \lambda x.A \rangle f$:

$$\sigma \models_w \langle \lambda x.A \rangle f \qquad \text{iff} \qquad \sigma(f)(w) \text{ is defined and } \sigma^{x \rhd \sigma(f)(w)} \models_w A$$

If $\sigma(f)(w)$ is not defined, then $\sigma \not\models_w \langle \lambda x.A \rangle f$; $\sigma \models_w \mathsf{D}f$ iff $\sigma(f)(w)$ is defined.

A formula A is *true in a world* w, $\models^{\mathcal{M}}_w A$, iff for all σ, $\sigma \models^{\mathcal{M}}_w A$; A is *true in a model*, $\models^{\mathcal{M}} A$, iff for all w, $\models^{\mathcal{M}}_w A$; A is *valid*, $\models A$, iff for all $\mathcal{M}$, $\models^{\mathcal{M}} A$.

3 Axiomatic system

Definition 3.1 FOIL.B is defined by the following axioms and rules [2]:

(i) All propositional tautologies
(ii) $\Box(A \to B) \to (\Box A \to \Box B)$
(iii) $\langle \lambda x.A \to B \rangle f \to (\langle \lambda x.A \rangle f \to \langle \lambda x.B \rangle f)$
(iv) $\langle \lambda x.A \rangle f \to A$, x not free in A
(v) $\langle \lambda x.A \rangle f \to \langle \lambda y.A[y/x] \rangle f$, y free for x in A
(vi) $\mathsf{D}f \to (\langle \lambda x.A \rangle f \vee \langle \lambda x.\neg A \rangle f)$
(vii) $x = x$
(viii) $x = y \to (P[x/z] \to P[y/z])$, P atomic formula;
(ix) $x = y \to \Box(x = y)$
(x) $x \neq y \to \Box(x \neq y)$
(xi) $\mathsf{D}f \to \langle \lambda y.\langle \lambda x.x = y \rangle f \rangle f$
(B) $A \to \Box\Diamond A$

$$\frac{A \quad A \to B}{B}\ MP \qquad \frac{A}{\Box A}\ N \qquad \frac{A \to B}{\langle \lambda x.A \rangle f \to \langle \lambda x.B \rangle f}\ \lambda\text{-reg}$$

Lemma 3.2 ([2, Proposition 4.1]) *Let* FOIL *be* FOIL.B *minus axiom* (B),

(i) $\vdash_{\mathsf{FOIL}} \mathsf{D}f \to (\neg\langle \lambda x.A \rangle f \leftrightarrow \langle \lambda x.\neg A \rangle f)$

(ii) $\vdash_{\mathsf{FOIL}} \langle \lambda x.A \rangle f \leftrightarrow (\mathsf{D}f \wedge A)$ provided x not free in A

(iii) $\vdash_{\mathsf{FOIL}} (\langle \lambda y.x = y \rangle f \wedge \langle \lambda y.z = y \rangle f) \to (x = z)$

Lemma 3.3 *The inductive set of rules* $CD(k)$*,* $k \in \mathbb{N}$*, is derivable in* FOIL.B*.*

$$\frac{A_0 \to \Box(A_1 \to \cdots \to \Box(A_k \wedge \mathsf{D}f \to \langle \lambda x.x \neq y \rangle f) \dots)}{A_0 \to \Box(A_1 \to \cdots \to \Box(A_k \to \neg \mathsf{D}f) \dots)}\ \text{CD(k)},\ k \in \mathbb{N},\ y \text{ not free in } A_i$$

Proof. We first prove that the rule $CD(0)$ is derivable in FOIL.

(a)	$A_0 \wedge \mathsf{D}f \to \langle \lambda x.x \neq y \rangle f$	Assumption
(b)	$\langle \lambda y.A_0 \rangle f \to [\langle \lambda y, x.\top \rangle f.f \to \langle \lambda y, x.x \neq y \rangle f.f]$	Ax. (i), Def. D, λ-reg
(c)	$A_0 \wedge \mathsf{D}f \to [\mathsf{D}f \wedge \langle \lambda x.\top \rangle f \to \langle \lambda y, x.x \neq y \rangle f.f]$	Lemma 3.2(ii)
(d)	$A_0 \wedge \mathsf{D}f \to [\mathsf{D}f \wedge \mathsf{D}f \to \langle \lambda y, x.x \neq y \rangle f.f]$	Def. D
(e)	$A_0 \to [\mathsf{D}f \to \langle \lambda y, x.x \neq y \rangle f.f]$	Axiom (i)
(f)	$A_0 \to [\mathsf{D}f \to \neg\langle \lambda y, x.x = y \rangle f.f]$	Lemma 3.2(i)
(g)	$A_0 \to [\mathsf{D}f \to \langle \lambda y, x.x = y \rangle f.f]$	Axiom (xi)
(h)	$A_0 \to \neg \mathsf{D}f$	From (f) and (g)

As is well known, the following rules are derivable from axiom B:

$$\frac{\Diamond A \to B}{A \to \Box B}\ DRB \qquad \frac{A \to \Box B}{\Diamond A \to B}\ DRB'\ .$$

CD(2) is derivable by the help of these rules:

(a)	$A_0 \to \Box(A_1 \to \Box(A_2 \wedge \mathsf{D}f \to \langle \lambda x.x \neq y \rangle f))$	Assumption
(b)	$(\Diamond A_0 \wedge A_1) \to \Box(A_2 \wedge \mathsf{D}f \to \langle \lambda x.x \neq y \rangle f)$	DRB'+ axiom (i)
(c)	$\Diamond(\Diamond A_0 \wedge A_1) \wedge A_2 \to (\mathsf{D}f \to \langle \lambda x.x \neq y \rangle f)$	DRB'+ axiom (i)
(d)	$\Diamond(\Diamond A_0 \wedge A_1) \wedge A_2) \to \neg \mathsf{D}f$	CD(0)
(e)	$(\Diamond A_0 \wedge A_1) \to \Box(A_2 \to \neg \mathsf{D}f)$	Axiom (i)+DRB
(j)	$A_0 \to \Box(A_1 \to \Box(A_2 \to \neg \mathsf{D}f))$	Axiom (i)+DRB

Analogously CD(k) is derivable for all $k \in \mathbb{N}$.

□

4 Completeness

We prove strong completeness by the usual Henkin-style technique, cf. [1]. Let P be a denumerable set of fresh object variables (to be called *parameters*) and let $\mathcal{L}^P$ be the language obtained by adding the set P to $\mathcal{L}$ and by imposing that parameters cannot be bound by λ. We use L (L^P) to the logic FOIL.B over the language $\mathcal{L}$ ($\mathcal{L}^P$, respectively), and Δ for a set of $\mathcal{L}^{(P)}$-formulas.

Definition 4.1
- Δ is L^P-*consistent* iff $\Delta \nvdash_{\mathsf{L}^P} \bot$.
- Δ is $\mathcal{L}^P$-*complete* iff for all $A \in \mathcal{L}^P$, either $A \in \Delta$ or $\neg A \in \Delta$.
- Δ is $\Diamond^k$-P-*rich* iff if $A_0 \wedge \Diamond(A_1 \wedge \cdots \wedge \Diamond(A_k \wedge \mathsf{D}f) \dots) \in \Delta$ then $A_0 \wedge \Diamond(A_1 \wedge \cdots \wedge \Diamond(A_k \wedge \mathsf{D}f \wedge \langle \lambda x(x = p)\rangle f) \dots) \in \Delta$ for some $p \in P \cup OBJ$.
- Δ is $\Box^k$-P-*inductive* iff $A_0 \to \Box(A_1 \to \cdots \to \Box(A_k \wedge \mathsf{D}f \to \langle \lambda x.x \neq p \rangle f))$ $\in \Delta$ for all $p \in P \cup OBJ$ only if $A_0 \to \Box(A_1 \to \cdots \to \Box(A_k \to \neg \mathsf{D}f) \dots) \in \Delta$.

- Δ is $\mathcal{L}^P$-*saturated* iff it is L^P-consistent, $\mathcal{L}^P$-complete, and $\Diamond^k$-P-rich ($\forall k \in \mathbb{N}$).

Lemma 4.2 *If Δ is $\mathcal{L}^P$-saturated, then it is $\Box^k$-P-inductive for all $k \in \mathbb{N}$.*

Lemma 4.3 (Lindenbaum-Henkin) *If Δ is an L-consistent set of formulas of $\mathcal{L}$, then there is an $\mathcal{L}^P$-saturated set $\Delta^\star$, for some denumerable set of parameters P, such that $\Delta^\star \supseteq \Delta$.*

Lemma 4.4 (Diamond-lemma for L) *If w is an $\mathcal{L}^P$-saturated set of formulas and $\Diamond A \in w$ then there is a set v of $\mathcal{L}^P$-formulas such that:*

(i) *v is $\mathcal{L}^P$-saturated;*

(ii) *$A \in v$;*

(iii) *$v \supseteq \Box^-(w)$, where $\Box^-(w) = \{A : \Box A \in w\}$;*

(iv) *for each $a \in P \cup OBJ$, $[a]_w = [a]_v$, where $[a]_w = \{b : a = b \in w\}$;*

Proof. Let $B_0, B_1, B_2, \dots B_n, B_{n+1} \dots$ be an enumeration of all $\mathcal{L}^P$-formulas.

- $\Delta_0 = \Box^-(w) \cup \{A\}$;
- Given Δ_n and B_n, we define Δ_{n+1}:
  (i) If $\Delta_n \cup \{B_n\}$ is not L^P-consistent, let $\Delta_{n+1} = \Delta_n \cup \{\neg B_n\}$;
  (ii) If $\Delta_n \cup \{B_n\}$ is L^P-consistent, we distinguish two cases:
    (a) If $B_n \equiv A_0 \wedge \Diamond(A_1 \wedge \cdots \wedge \Diamond(A_k \wedge \mathsf{D}f)\dots)$ for some $A_0, \dots, A_k$, let $\Delta_{n+1} = \Delta_n \cup \{A_0 \wedge \Diamond(A_1 \wedge \cdots \wedge \Diamond(A_k \wedge \mathsf{D}f \wedge \langle\lambda x.x = p\rangle f)\dots)\}$ for some $p \in P \cup OBJ$ such that the resulting set is L^P-consistent;
    (b) Else, $\Delta_{n+1} = \Delta_n \cup \{B_n\}$.

Lemma 4.5 *Each element of the chain $\Delta_0, \Delta_1, \dots, \Delta_n, \dots$ is L^P-consistent.*

Proof. Δ_0 is L^P-consistent by modal reasoning. Assume, by induction hypothesis, that Δ_n is L^P-consistent. We consider only case (ii)(a).

Suppose by *reductio* that there is no $p \in P \cup OBJ$ such that the set $\Delta_n \cup \{A_0 \wedge \Diamond(A_1 \wedge \cdots \wedge \Diamond(A_k \wedge \mathsf{D}f \wedge \langle\lambda x.x = p\rangle f)\}$ is L^P-consistent. Then, for all $p \in P$, $\Delta_n \vdash_{\mathsf{L}^P} (A_0 \wedge \Diamond(A_1 \wedge \cdots \wedge \Diamond(A_k \wedge \mathsf{D}f \wedge \langle\lambda x.x = p\rangle f)) \to \bot$. By modal reasoning $\Delta_n \vdash_{\mathsf{L}^P} (A_0 \to \Box(A_1 \to \cdots \to \Box(A_k \wedge \mathsf{D}f \to \neg\langle\lambda x.x = p\rangle f))$. By Lemma 3.2(i), $\Delta_n \vdash_{\mathsf{L}^P} (A_0 \to \Box(A_1 \to \cdots \to \Box(A_k \wedge \mathsf{D}f \to \langle\lambda x.x \neq p\rangle f))$.

Moreover, Δ_n is just $\Box^-(w) \cup \{C_1, \dots C_m\}$ for some finite set of formulas $\{C_1, \dots C_m\}$, therefore, where $C \equiv C_1 \wedge \cdots \wedge C_m$,
$\Box^-(w) \vdash_{\mathsf{L}^P} C \wedge A_0 \to \Box(A_1 \to \cdots \to \Box(A_k \wedge \mathsf{D}f \to \langle\lambda x.x \neq p\rangle f))$ for all $p \in P \cup OBJ$. Thus $w \vdash_{\mathsf{L}^P} \Box(C \wedge A_0 \to \Box(A_1 \to \cdots \to \Box(A_k \wedge \mathsf{D}f \to \langle\lambda x.x \neq p\rangle f))$ for all $p \in P \cup OBJ$, and, for all $p \in P \cup OBJ$,
$w \vdash_{\mathsf{L}^P} \top \to \Box(C \wedge A_0 \to \Box(A_1 \to \cdots \to \Box(A_k \wedge \mathsf{D}f \to \langle\lambda x.x \neq p\rangle f)))$.
Since w is $\mathcal{L}^P$-saturated, by lemma 4.2, w is $\Box^j$-P-inductive for all $j \in \mathbb{N}$, hence, in particular w is $\Box^{k+1}$-P-inductive, therefore
$w \vdash_{\mathsf{L}^P} \top \to \Box(C \wedge A_0 \to \Box(A_1 \to \cdots \to \Box(A_k \to \neg\mathsf{D}f)))$.
It follows that $w \vdash_{\mathsf{L}^P} \Box(C \wedge A_0 \to \Box(A_1 \to \cdots \to \Box(A_k \to \neg\mathsf{D}f)))$,
$(C \wedge A_0 \to \Box(A_1 \to \cdots \to \Box(A_k \to \neg\mathsf{D}f))) \in \Box^-(w)$
$\Delta_n \vdash_{\mathsf{L}^P} A_0 \to \Box(A_1 \to \cdots \to \Box(A_k \to \neg\mathsf{D}f))$,
But this contradicts the L^P-consistency of $\Delta_n \cup \{B_n\}$. $\Box$

Let $v = \bigcup_{n\in\mathbb{N}} \Delta_n$. The set v is L^P-consistent and satisfies all the properties of the lemma. □

Definition 4.6 Let us consider the frame $\langle G^{\mathsf{L}}, R\rangle$ where:
• G^{L} is the class of all $\mathcal{L}^P$-saturated sets of formulas of $\mathcal{L}^P$ for some denumerable set of parameters P; • wRv iff $\Box^-(w) \subseteq v$.

This frame is likely to be composed of a number of parts, each completely isolated from any of the others. Following [4, p. 78], a *cohesive* frame is one in which, for every $w, w' \in W^{\mathsf{L}}$, $w(R \cup R^{-1})^n w'$ for some $n \geq 0$.

Definition 4.7 [Normal canonical model] A *normal canonical model* for L is a tuple $\mathcal{M}^{\mathsf{L}} = \langle W^{\mathsf{L}}, R, D_O, D_L, D_I, V\rangle$, where:

- $\langle W^{\mathsf{L}}, R\rangle$ is any of the cohesive frames of which $\langle G^{\mathsf{L}}, R\rangle$ is composed;
- $D_O = \{[a]_w : \text{for some } w \in W^{\mathsf{L}}, \text{ where } a \in OBJ \cup P\}$;
- $D_L = \{\ell_{\widehat{f}} : f \in INT\}$;
- $D_I = \{\widehat{f} : \ell_{\widehat{f}} \in D_L\}$, where, for all $w \in W^{\mathsf{L}}$, if $\mathsf{D}f \in w$, then, for some $[p] \in D_O$ such that $\langle \lambda y(y = p\rangle f \in w\}$, $\widehat{f}(w) = \langle [p], \ell_{\widehat{f}}\rangle$; else $\widehat{f}(w) = \ell_{\widehat{f}}$.
- the valuation V is a function with domain W^{L} that is such that:
  $V(P^{n,m}, w) = \{\langle [a_1]_w, ..., [a_n]_w, \widehat{f_1}, \dots, \widehat{f_m}\rangle : Pa_1, ..., a_n, f_1, \dots, f_m \in w\}$.
  $V(=^{2,0}, w) = \{\langle [a], [a]\rangle : \text{where } a \in OBJ \cup P\ \}$.

Remark 4.8 D_O is well defined because the frame $\langle W^{\mathsf{L}}, R\rangle$ is cohesive and so for every $w, v \in W^{\mathsf{L}}$, $[a]_w = [a]_v$, in fact $\{b : (a = b) \in w\} = \{b : (a = b) \in v\}$ thanks to axioms (ix) and (x). So we can write $[a]$ instead of $[a]_w$.

D_I is well defined because if $\mathsf{D}f \in w$ there is at least a $p \in P \cup OBJ$ such that $\langle \lambda y.y = p\rangle f \in w$ since w is $\Diamond^0$-P-rich, moreover such a $[p]$ is unique, in fact, by Lemma 3.2(iii), $\vdash_{\mathsf{L}} [\langle \lambda y.p = y\rangle f \wedge \langle \lambda y.p' = y\rangle f] \rightarrow (p = p')$.

Moreover, we avoid the problem noted in [3] of mapping two distinct intensional variables satisfying different formulas to the same intension: if $f \not\equiv g$, then $\ell_f \neq \ell_g$ and, therefore, f and g will be assigned to different intensions even if $(\langle \lambda x.x = p\rangle f \in w)$ iff $(\langle lx.x = p\rangle g \in w)$, for all $w \in W^{\mathsf{L}}$.

Lemma 4.9 (Truth lemma) *Let $\mathcal{M}^{\mathsf{L}}$ be a normal canonical model for L and let σ be the* canonical assignment *such that $\sigma(a) = [a]$ and $\sigma(f) = \widehat{f}$. For all $w \in W^{\mathsf{L}}$ and for all formula A of $\mathcal{L}^P$,* $\sigma \models^{\mathcal{M}^{\mathsf{L}}}_w A$ *iff* $A \in w$.

Theorem 4.10 *Any* FOIL.B*-consistent set of formulas is satisfied (under the can. ass.) in some world of a symmetric canonical model with constant domain.*

References

[1] Corsi, G., *A unified completeness theorem for quantified modal logics*, The Journal of Symbolic Logic **67** (2002), pp. 1483–1510.
[2] Fitting, M., *FOIL axiomatized*, Studia Logica **84** (2006), pp. 1–22.
[3] Fitting, M., *Correction to 'FOIL axiomatized'*, Studia Logica **85** (2007), p. 275.
[4] Hughes, G. E. and M. J. Cresswell, "A Companion to Modal Logic," Methuen, 1984.

Hybrid Logic in the Isabelle Proof Assistant: Benefits, Challenges and the Road Ahead

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Abstract

We outline benefits of formalizing a proof system for hybrid logic in the proof assistant Isabelle/HOL, showcase how the process of formalization can shape our proofs, and describe our current work on formalizing completeness of a more restrictive system. Formalization: `https://devel.isa-afp.org/entries/Hybrid_Logic.html`

Keywords: Hybrid logic, Seligman-style tableau, Isabelle/HOL

1 Introduction

Basic hybrid logic extends ordinary modal logic with nominals, a special sort of propositional symbol true at exactly one world, and satisfaction statements, $@_i\phi$, which are true if and only if the formula ϕ is true in the world named by nominal i. The well-formed formulas of the basic hybrid logic are defined as follows, where x is a propositional symbol and we use i, j, k, a, b for nominals:

$$\phi, \psi ::= x \mid i \mid \neg\phi \mid \phi \vee \psi \mid \Diamond\phi \mid @_i\phi$$

The language is interpreted on Kripke models $\mathfrak{M}$, consisting of a frame (W, R) and a valuation of propositional symbols V. Here W is a non-empty set of worlds and R is a binary accessibility relation between them. To interpret nominals we use an assignment g mapping them to elements of W; if $g(i) = w$ we say that nominal i denotes w. Formula satisfiability is defined as follows:

$$\begin{array}{lll}
\mathfrak{M}, g, w \models x & \text{iff} & w \in V(x) \\
\mathfrak{M}, g, w \models i & \text{iff} & g(i) = w \\
\mathfrak{M}, g, w \models \neg\phi & \text{iff} & \mathfrak{M}, g, w \not\models \phi \\
\mathfrak{M}, g, w \models \phi \vee \psi & \text{iff} & \mathfrak{M}, g, w \models \phi \text{ or } \mathfrak{M}, g, w \models \psi \\
\mathfrak{M}, g, w \models \Diamond\phi & \text{iff} & \text{for some } w',\ wRw' \text{ and } \mathfrak{M}, g, w' \models \phi \\
\mathfrak{M}, g, w \models @_i\phi & \text{iff} & \mathfrak{M}, g, g(i) \models \phi
\end{array}$$

We have just presented basic hybrid logic using (semi-formal) natural language, but we could have presented it using a proof assistant like Isabelle/HOL [7] instead. This forces us to be more precise: we would have to define hybrid logic in the proof assistant's logic (here, higher-order logic). But we can then do our metatheory in higher-order logic and machine check its correctness. This leaves no room for ambiguity or mistakes since every statement compiles to the primitives of the proof assistant (that we trust to be correct). Of course, we will have to supply more proof detail which can result in more verbose proofs; nonetheless, used skillfully, formalization can help guide our exploration of metatheory, and suggest new ideas, as we hope to show.

Hybrid logic has received little such treatment. Doczkal and Smolka formalize hybrid logic with nominals but no satisfaction operators in constructive type theory using the proof assistant Coq. They give algorithmic proofs of small model theorems and computational decidability of satisfiability, validity, and equivalence of formulas [3]. In Isabelle/HOL, Linker formalizes the semantic embedding of a spatio-temporal multi-modal logic that includes a hybrid logic-inspired at-operator but has no proof system [6]. The present work is the first sound and complete formalized proof system for hybrid logic that we know of. We have briefly described an earlier version of the formalization in a short paper for an automated reasoning audience [4], but that paper did not cover the notion of "potential" for restricting the GoTo rule.

2 Seligman-Style Tableau System

The proof system must handle the fact that a hybrid logic formula is true relative to a given world. Figures 1a and 1b depict two strategies for this.

(a)	(b)
$\vdots$	$\vdots$
	———
	i
$@_i\phi_1$	ϕ_1
$@_i\phi_2$	ϕ_2
$\vdots$	$\vdots$
	———
	j
$@_j\psi_1$	ψ_1
$\vdots$	$\vdots$

0.	a		
1.	$\neg(\neg @_i\phi \vee @_i\phi)$		[0]
2.	$\neg\neg @_i\phi$	$(\neg\vee)$ 1	[1]
3.	$\neg @_i\phi$	$(\neg\vee)$ 1	[2]
4.	$@_i\phi$	$(\neg\neg)$ 2	[3]
5.	i	GoTo	[2]
6.	$\neg\phi$	$(\neg @)$ 3	[3]
7.	ϕ	$(@)$ 4	[4]
	$\times$		

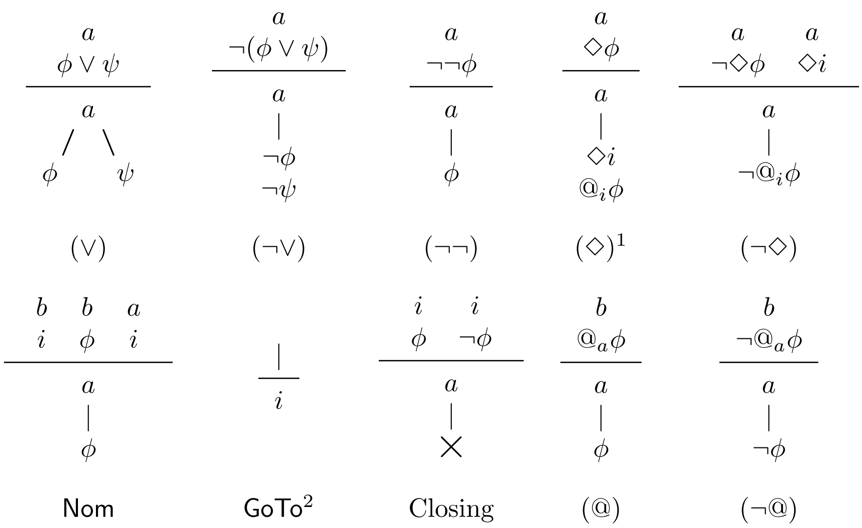
(a) Internalized. (b) Seligman-style. (c) Seligman-style tableau example.

Fig. 1. Tableau styles. (c) displays potential in the fourth column.

Internalized tableau systems work exclusively with satisfaction statements while the Seligman-style tableau system handles arbitrary formulas, giving a more local proof style, by dividing branches into blocks of formulas that are all true at the same world. Each pair of blocks is separated by a horizontal line and every block starts with a nominal dubbed the opening nominal, denoting that world. We call a block with opening nominal i an "i-block."

Figure 2 gives the tableau rules. Every rule has input formulas above the vertical line(s) and output below. The output of GoTo is a new block with corresponding opening nominal, while the other rules extend the last, so-called "current" block. When a rule has multiple input formulas we write them next to each other. Above each input formula, we write the opening nominal of the block it occurs on. Similarly, the opening nominal of the current block is the first thing below the horizontal line. Any formula on the current block may be used as input under the same restrictions on opening nominals. The system resembles (and simplifies) the one developed by Blackburn et al. [1], notably by having single-input (@) and (¬@) rules and assuming that all blocks have an opening nominal causing us to omit a rule.

Figure 1c gives an example tableau for the formula $\neg @_i \phi \vee @_i \phi$ which is negated and placed on a block with an arbitrary opening nominal. Note how the GoTo rule switches perspective to the world denoted by i while consuming a unit of potential in the fourth column.



[1] i is fresh, ϕ is not a nominal.
[2] i is not fresh.

Fig. 2. Tableau rules

We formalize this proof system as an inductive predicate, $\vdash$, in Isabelle by specifying for which branches $\vdash$ holds. For example, the closing condition becomes the following code that allows you to close any branch where, for some p and i, both p and $\neg p$ occur on i-blocks ("at i") in the branch:

Close: ⟨*p at i in branch* $\Longrightarrow$ (¬ *p*) *at i in branch* $\Longrightarrow$ *n* $\vdash$ *branch*⟩

Here, n is the "potential" from Figure 1c. After defining all cases we can type in a closing tableau and have the computer check that every rule is applied according to our definition: we get a proof checker for free. Moreover, we can machine verify proofs of soundness and completeness.

3 Rule Induction

When we define the proof system, Isabelle provides a principle for proving statements by induction on the construction of a closing tableau. We consider a special case of the principle here, which is used to show lemmas of the form "if the branch Θ closes then so does $f(\Theta)$" where f is some transformation of the branch. Examples of transformations could be to rename nominals or to omit redundant occurrences of formulas.

The induction principle then instructs us, for each rule, to assume that the branch extended by that rule's output has a closing tableau when transformed and show that a closing tableau exists without the extension, typically by applying the rule in question. For instance, in the $(\neg\neg)$ case we assume, first, the premise of the rule, that $\neg\neg\phi$ occurs on an a-block in Θ where a is the opening nominal of the current block. Second: we assume as induction hypothesis that the transformation of Θ extended by ϕ has a closing tableau. To prove the case we need to show that the transformation of just Θ has a closing tableau.

This induction principle is our motivation for rephrasing the following restriction on the proof system by Blackburn et al. [1]:

Original R4 The GoTo rule cannot be applied twice in a row.

Current R4 The GoTo rule consumes one potential. The remaining rules add one potential and we are allowed to start from any amount of potential.

1.	a	
2.	$\neg\neg\phi$	
	⋮	
3.	a	GoTo
4.	ϕ	$(\neg\neg)$ 2
5.	i	GoTo

(a) Starting point.

1.	a	
2.	$\neg\neg\phi$	
3.	ϕ	
	⋮	
4.	a	GoTo
5.	ϕ	~~$(\neg\neg)$ 2~~
6.	i	~~GoTo~~

(b) Transformed. 5 and 6 are now illegal.

Fig. 3. Unjustified GoTo after weakening on line 3. We assume restriction **R1** [1], that extensions must be new.

The original restriction rules out infinite branches that consist of repeated applications of GoTo. Potential does the same because it decreases with each application. This new formulation, however, works better with the induction principle outlined above, since that principle may force us to apply GoTo twice in a row. Consider Figure 3b where the transformation of the branch means we should not apply GoTo on line 4 as in the tableau we are mimicking but go directly to line 6. With the original **R4** we would need a more intricate transformation of the branch (or a weaker lemma), but with the current restriction we can simply assume that we start with more potential, making the detour benign. The restriction preserves completeness as any closed tableau is finite.

Also, we can always start from a single unit:

Theorem 3.1 (Potential) *If a branch can be closed then it can be closed starting from a single unit of potential. (cf. "No detours" in the formalization.)*

4 Current Work

We have lifted equivalents of the four relevant restrictions by Blackburn et al. [1] (**R1**, **R2** and **R5**) in previous work [4]. Unfortunately, the Nom rule as given can still be used to construct infinite branches [1]. Blackburn et al. replace it with a three-part Nom* rule without this problem and show that it is sufficient for their translation-based completeness proof [1]. Instead of splitting it, we may impose the following, equivalent restriction on the general Nom rule:

Nom* $i = a$ and ϕ is not k or $\Diamond k$ for any k introduced by the ($\Diamond$) rule.

This restriction means that "($\Diamond$)-produced" nominals can only appear on their own as opening nominals. This breaks a symmetry otherwise present in exhausted branches: if nominal i appears on a k-block then k also appears on an i-block. The synthetic completeness proof by Jørgensen et al. [5] that we have previously formalized [4] makes use of this symmetry in their modeling of open exhausted branches and their model existence result. We have overcome this by (a) updating the definition of Hintikka sets to model our non-symmetric branches and (b) applying the model existence result by Bolander and Blackburn for a terminating internalized calculus [2] to our synthetic setting.

5 Conclusion

Modern proof assistants are more than capable of handling non-trivial proof systems and their metatheory. It can still be beneficial to shape our proofs such that they work well with the tools provided by the assistant, but in return we gain precision and absolute trust in the correctness of our results.

References

[1] Blackburn, P., T. Bolander, T. Braüner and K. F. Jørgensen, Completeness and Termination for a Seligman-style Tableau System, Journal of Logic and Computation **27** (2017), pp. 81–107.

[2] Bolander, T. and P. Blackburn, Termination for Hybrid Tableaus, Journal of Logic and Computation **17** (2007), pp. 517–554.

[3] Doczkal, C. and G. Smolka, Constructive Formalization of Hybrid Logic with Eventualities, in: Certified Programs and Proofs (CPP). Proceedings, 2011, pp. 5–20.

[4] From, A. H., P. Blackburn and J. Villadsen, Formalizing a Seligman-style Tableau System for Hybrid Logic, in: N. Peltier and V. Sofronie-Stokkermans, editors, Automated Reasoning (2020), pp. 474–481.

[5] Jørgensen, K. F., P. Blackburn, T. Bolander and T. Braüner, Synthetic Completeness Proofs for Seligman-style Tableau Systems, in: Advances in Modal Logic, Volume 11, 2016, pp. 302–321.

[6] Linker, S., Hybrid Multi-Lane Spatial Logic, Archive of Formal Proofs (2017), `http://isa-afp.org/entries/Hybrid_Multi_Lane_Spatial_Logic.html`, Formal proof.

[7] Nipkow, T., L. C. Paulson and M. Wenzel, "Isabelle/HOL - A Proof Assistant for Higher-Order Logic," Lecture Notes in Computer Science **2283**, Springer, 2002.

FOLTL with counting quantifiers over finite timelines with expanding domains is Ackermann-complete

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Abstract

This paper addresses a gap in the literature concerning the precise complexity of the satisfiability problem for the one-variable fragment of first-order linear temporal logic (FOLTL) with arbitrary *counting quantifiers* $\mathsf{FOLTL}^{\#}_{\mathsf{fin}}$ over expanding domain models. By exploiting explicit bounds on Dickson's Lemma, we obtain an Ackermannian upper-bound on the size of satisfying models for $\mathsf{FOLTL}^{\#}_{\mathsf{fin}}$ over expanding domains, yielding an optimal Ackermann-time decision procedure.

Keywords: first-order temporal logic, counting quantifiers, Dickson's Lemma, Ackermann-complete.

1 Preliminaries

1.1 First-order temporal logics with counting quantifiers

In what follows, we shall consider the one-variable fragment of the first-order linear temporal language comprising a countably infinite set of (monadic) predicate symbols $\mathsf{Pred} = \{P_0, P_1, \dots\}$ and sole first-order variable x. We denote by $\mathcal{QTL}^{\#}_1$ the set of all FOLTL formulas with *counting quantifiers* defined by the following grammar:

$$\varphi \ ::= \ P_i(x) \ | \ \neg\varphi \ | \ (\varphi_1 \wedge \varphi_2) \ | \ \mathsf{F}\,\varphi \ | \ \mathsf{X}\,\varphi \ | \ \exists_{\leq c} x\,\varphi$$

where $P_i \in \mathsf{Pred}$ and $c \in \mathbb{N}$ specifies the *capacity* of the quantifier $\exists_{\leq c}$. Other Boolean connectives can be defined in the usual way, together with temporal operator $\mathsf{G}\,\varphi := \neg\mathsf{F}\,\neg\varphi$, and quantifiers $\exists x\,\varphi := \neg\exists_{\leq 0} x\,\varphi$ and $\forall x\,\varphi := \neg\exists x\,\neg\varphi$.

Formulas of $\mathcal{QTL}^{\#}_1$ are interpreted in *first-order Kripke models* of the form $\mathfrak{M} = \langle \mathfrak{T}, D, \mathfrak{d}, \mathcal{I}\rangle$, where $\mathfrak{T}$ is an initial segment of the natural number under their usual ordering, D is a non-empty set of *domain objects* and $\mathfrak{d}$ is a *domain function* which associates each instance $k \in \mathfrak{T}$ with a non-empty subset $\mathfrak{d}(k) \subseteq D$. Finally, $\mathcal{I} : \alpha \times \mathsf{Pred} \to 2^D$ is a function associating each $k \in \mathfrak{T}$ and each predicate symbol $P_i \in \mathsf{Pred}$ with a subset $\mathcal{I}(k, P_i) \subseteq \mathfrak{d}(k)$.

We say that a model is *expanding* in the case that $\mathfrak{d}(n) \subseteq \mathfrak{d}(m)$, whenever $n < m$, respectively. Satisfiability is defined in the usual way, with $\mathfrak{M}, k \models^a \mathsf{X}\,\varphi$ iff $\mathfrak{M}, (k+1) \models^a \varphi$ and $\mathfrak{M}, \ell \models^a \mathsf{F}\,\varphi$ iff $\mathfrak{M}, \ell \models^a \varphi$, for some $\ell > k$. Counting quantifiers are interpreted so that $\mathfrak{M}, k \models^a \exists_{\leq c} x\, \varphi$ iff $\left|\{b \in \mathsf{dom}\,(k) : \mathfrak{M}, k \models^b \varphi\}\right| \leq c$.

In what follows, we are concerned with the following decision problem:

$\underline{\mathsf{FOLTL}^{\#}_{\mathsf{fin}}\text{-SAT:}}$

Input: Given a formula $\varphi \in \mathcal{QTL}_1^{\#}$,

Question: Is there a first-order expanding model $\mathfrak{M} = \langle \mathfrak{T}, D, \mathfrak{d}, \mathcal{I} \rangle$ such that $\mathfrak{T}$ is finite and $\mathfrak{M}, 0 \models^a \varphi$ for some $a \in \mathfrak{d}(0)$?

If we were to consider satisfiability with respect to *constant* domain models then the satisfiability problem is known to be non-recursively enumerable, even if we were to restrict the language to the X-free fragment with sole quantifiers $\{\exists_{\leq 0}, \exists_{\leq 1}\}$ [6].

In what follows, the size of $\varphi \in \mathcal{QTL}_1^{\#}$, denoted $\|\varphi\|$, is taken to be the number of symbols it comprises with the capacity of counting quantifiers encoded in binary, so that $\mathsf{cap}(\varphi) < \log_2(\|\varphi\|)$ and $|\mathsf{sub}(\varphi)| < \|\varphi\|$, where $\mathsf{cap}(\varphi)$ denotes the maximum capacity appearing in φ and $\mathsf{sub}(\varphi)$ denotes the set of subformulas of φ.

1.2 The fast-growing hierarchy

For each countable ordinal $\alpha \in \mathrm{Ord}$ we define the function $F_\alpha : \mathbb{N} \to \mathbb{N}$ by taking

$$F_0(n) := n + 1, \qquad F_{\alpha+1}(n) := F_\alpha^n(n), \qquad \text{and} \qquad F_\lambda(n) := F_{\lambda(n)}(n) \qquad (\dagger)$$

if $\lambda \in \mathrm{Ord}$ is a countable limit ordinal, where $\lambda(n)$ is a *fundamental sequence* for λ (see [9] for details). For our purposes, it is enough to note that we obtain a version of Ackermann's function $F_\omega(n) = F_n(n)$ by a *diagonalization* of the sequence $F_0, F_1, F_2, \ldots$ [1]. For each countable ordinal $\alpha \in \mathrm{Ord}$, we define the complexity class $\mathbf{F}_\alpha$ to be the set of all decision problems that can be solved by a (deterministic) Turing Machine in time bounded by some fast-growing function F_α of some function $p(n) \in O(F_\beta(n))$ for $\beta < \alpha$, where n is the size of the input.

2 Result

Definition 2.1 Let $\mathsf{Types}(\varphi) \subseteq 2^{\mathsf{sub}(\varphi)}$ denote the set all Boolean saturated set of subformulas of φ. We define a *quasistate* for φ to be a pair (T, μ) such that:

(qs1) $T \subseteq \mathsf{Types}(\varphi)$ is a non-empty set of *types* for φ,

(qs2) $\mu : T \to \{1, \ldots, \mathsf{cap}(\varphi), \mathsf{cap}(\varphi)+1\}$ is a *'multiplicity'* function,

(qs3) *($\exists_{\leq c}$-saturation)* For all $t \in T$ and $(\exists_{\leq c} x\, \xi) \in \mathsf{sub}(\varphi)$, we have that $(\exists_{\leq c} x\, \xi) \in t$ iff $\sum\{\mu(t') : \xi \in t' \text{ and } t' \in T\} \leq c$.

Definition 2.2 A *quasimodel for* φ is a tuple $\mathfrak{Q} = \langle N, \boldsymbol{q}, I, \mathfrak{R}\rangle$ such that:

(qm1) $N \in \mathbb{N}$, $\boldsymbol{q}$ is a function associating each $k < N$ with a quasistate $\boldsymbol{q}(k) = (T_k, \mu_k)$, and $\mathfrak{R}$ is an set of partial functions (called *runs*) r_i, indexed with indices from I, such that $r_i(k) \in T_k$ for each $k \in \mathsf{dom}\,(r_i)$, where $\mathsf{dom}\,(r_i)$ denotes the domain over which r_i is defined.

(qm2) There is some $i \in I$ such that $\varphi \in r_i(0)$,

(qm3) *(expanding)* For all $i \in I$, if $k \in \mathsf{dom}\,(r_i)$ and $k < k'$ then $k' \in \mathsf{dom}\,(r_i)$.

(qm4) *(X-coherence)* For all $i \in I$, $k \in \mathsf{dom}\,(r_i)$ and $\mathsf{X}\,\xi \in \mathsf{sub}(\varphi)$,

$$\mathsf{X}\,\xi \in r_i(k) \quad\Longleftrightarrow\quad \xi \in r_i(k+1),$$

(qm5) *(F-coherence)* For all $i \in I$, $k \in \mathsf{dom}\,(r_i)$ and $\mathsf{F}\,\xi \in \mathsf{sub}(\varphi)$,

$$\mathsf{F}\,\xi \in r_i(k) \quad\Longleftrightarrow\quad \xi \in r_i(k') \quad \text{for some } k' > k,$$

(qm6) For all $k \in W$ and $t \in T_k$, we have $\mu_k(t) = \min\big(\left|I(k,t)\right|, \mathsf{cap}(\varphi)+1\big)$,

where $I(k,t) = \{i \in I : k \in \mathsf{dom}\,(r_i) \text{ and } r_i(k) = t\}$ denotes the set of indices of runs passing through type $t \in T_k$ of quasistate $\boldsymbol{q}(k)$.

For each $k < N$, we define the *signature* of $\boldsymbol{q}(k)$ to be the $\mathbb{N}^d$-vector:

$$\sigma(k) = \langle |I(k,t)| \;:\; t \in \mathsf{Types}(\varphi)\rangle \in \mathbb{N}^d$$

of dimension $d = |\mathsf{Types}(\varphi)| < 2^{\|\varphi\|}$. We say that $\mathfrak{Q}$ is *controlled* if it satisfied the additional condition that:

(ctrl) $\|\sigma(k)\|_\infty \leq 4^{(\|\varphi\|+k)}$, for all $k < N$, where $\|\boldsymbol{x}\|_\infty = \max_{i=1}^d x_i$ denotes the ∞-norm of $\boldsymbol{x} = \langle x_1, \ldots, x_d\rangle \in \mathbb{N}^d$,

and that $\mathfrak{Q}$ is *small* if it satisfies the additional condition that:

(sml) $N < F_{(d+2)}(\|\varphi\|)$, where $F_{(d+2)}$ is as defined in (†).

Lemma 2.3 *φ is $\mathsf{FOLTL}^{\#}_{\mathsf{fin}}$-satisfiable iff there is a quasimodel for φ.*

Proof. The proof is routine and follows similar constructions in [8,5]. □

Lemma 2.4 *If φ has a quasimodel then φ has a* controlled *quasimodel.*

Proof. Suppose that $\mathfrak{Q} = \langle N, \boldsymbol{q}, I, \mathfrak{R}\rangle$ is a quasimodel for φ. For each $k < N$, $t \in T_k$ and $m < \mu_k(t)$, we fix an index $i_{(k,t,m)} \in I$ such that *(i)* $k \in \mathsf{dom}\,\big(r_{i_{(k,t,m)}}\big)$ and $r_{i_{(k,t,m)}}(k) = t$, and *(ii)* If $m \neq m'$ then $i_{(k,t,m)} \neq i_{(k,t,m')}$.

Let I' be the set of all such indices, and define a new run r'_i for each $i = i_{(k,t,m)} \in I'$ by taking $\ell \in \mathsf{dom}\,(r'_i)$ iff $\ell \geq k$ and $r'_i(\ell) = r_i(\ell)$, for all $\ell \geq k$; that is to say that we trim the domains of the runs. Let $\mathfrak{R}'$ be the set of all such runs indexed by I'. Note that, since $\mu_k(t) \leq (\mathsf{cap}(\varphi)+1) \leq 2^{\|\varphi\|}$ and $|T_k| \leq 2^{\|\varphi\|}$, we have that $\sum_{t\in\mathsf{Types}(\varphi)} |I'(0,t)| \leq 4^{\|\varphi\|}$ and $\sum_{t\in\mathsf{Types}(\varphi)} |I'(k+1,t)| \leq 4^{\|\varphi\|} + \sum_{t\in\mathsf{Types}(\varphi)} |I'(k,t)|$, from which we deduce that

$$\|\sigma(k)\|_\infty \;\leq\; \sum_{t\in\mathsf{Types}(\varphi)} |I'(k,t)| \;\leq\; (k+1)\cdot 4^{\|\varphi\|} \;\leq\; 4^{(\|\varphi\|+k)}.$$

for all $k < N$, since $(k+1) \leq 4^k$. It is a routine exercise to show that $\mathfrak{Q}' = \langle N, \boldsymbol{q}, I', \mathfrak{R}' \rangle$ is a *controlled* quasimodel for φ, as required. Note, also, that if $\mathfrak{Q}$ is *small* then so too is $\mathfrak{Q}'$, since the size of the timeline remains unchanged. □

Lemma 2.5 *If φ has a quasimodel then φ has a* small *quasimodel.*

Proof. Suppose to the contrary that φ is satisfiable but does not have a *small* quasimodel. Let $\mathfrak{Q} = \langle N, \boldsymbol{q}, I, \mathfrak{R} \rangle$ be the smallest quasimodel for φ, which we may assume does not to satisfy **(sml)**, and so $N > F_{(d+2)}(\|\varphi\|)$. Moreover, without loss of generality, we may assume that $\mathfrak{Q}$ is a *controlled* quasimodel for φ, courtesy of Lemma 2.4. Consider the sequence of signatures

$$\Sigma = \left\langle \sigma(k) \in \mathbb{N}^d : k < \alpha \right\rangle$$

where $d < 2^{\|\varphi\|}$. Note that, by **(ctrl)**, we have that $\|\sigma(k)\|_\infty \leq 4^{(\|\varphi\|+k)}$ and so the sequence is $(4^n, \|\varphi\|)$-*controlled* in the sense of [3], where it is proved the the maximum length of any such *'bad'* sequence for which Dickson's Lemma [1] does not apply is at most $F_{(d+2)}(\|\varphi\|)$. However, since $N > F_{(d+2)}(\|\varphi\|)$, it then follows that there must be some $n < m < N$ such that $\sigma(n) \leq \sigma(m)$, which is to say that $|I(n,t)| \leq |I(m,t)|$, for all $t \in \mathsf{Types}(\varphi)$.

It follows that there is some family of injections $\eta_t : I(n,t) \to I(m,t)$, for $t \in T_n$, from which we can construct an injection $\eta : A \to I$ where $A = \{i \in I : n \in \mathsf{dom}\,(r_i)\} \subseteq I$. Let $B = I - \mathrm{rng}(\eta)$ denote all those indices that do not appear in the range of η. Now let $I' = \{(i,a) : i \in A\} \cup \{(i,b) : i \in B\}$ be the disjoint union of A and B.

We define a new quasimodel $\mathfrak{Q} = \langle N', \boldsymbol{q}', \mathcal{I}', \mathfrak{R}' \rangle$ by making an excision of the sub-interval $[n,m)$ and stitching together the runs bridging the cut, similar to the approach taken in [7]. To this end, let $\lambda : \omega \to \omega$ is a relabelling such that $\lambda(k) = k$, for $k < n$, and $\lambda(k) = k + m - n$, for $k \geq n$. Take $N' = (N - m + n)$ and $\boldsymbol{q}'(k) = \boldsymbol{q}(\lambda(k))$, for all $k < N'$. For each $(i,x) \in I'$, let $r'_{(i,x)}$ be a new run obtained by 'stitching' together runs indexed by $i \in A$ with those indexed by $\eta(B) \in I$ across the excision, by taking

$$r'_{(i,x)}(k) \;=\; \begin{cases} r_i(k) & \text{if } x = a \text{ and } k < n \\ r_{\eta(i)}(\lambda(k)) & \text{if } x = a \text{ and } k \geq n \\ r_i(\lambda(k)) & \text{if } x = b \text{ and } k \geq n \\ \text{undefined} & \text{if } x = b \text{ and } k < n. \end{cases}$$

Take $\mathfrak{R}'$ to the set set of all such runs indexed by I'. It is then straightforward to check that $\mathfrak{Q}'$ is a quasimodel for φ. Moreover, $\mathfrak{Q}'$ is smaller than $\mathfrak{Q}$, contrary to the supposition that $\mathfrak{Q}$ be the smallest such quasimodel. Hence, there must be some quasimodel satisfying **(sml)**. □

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[1] Dickson's Lemma states for every infinite sequence $\langle \boldsymbol{x}_i \in \mathbb{N}^d : i < \omega \rangle$ there are $i < j$ such that $\boldsymbol{x}_i \leq \boldsymbol{x}_j$. A *'bad'* sequence is any finite sequence without this property.

Theorem 2.6 $\mathsf{FOLTL}^{\#}_{\mathsf{fin}}$ *has the* F_ω*-bounded finite model property.*

Proof. By alternating between application of Lemmas 2.4 and 2.5 we have that φ is satisfiable if and only if it has a quasimodel that is both small and controlled. Lemma 2.3, then yields a model for φ whose size is bounded by $O(F_{(d+2)}(\|\varphi\|)) < O(F_\omega(\|\varphi\|))$, as required. □

Corollary 2.7 *The satisfiability problem for* $\mathsf{FOLTL}^{\#}_{\mathsf{fin}}$ *is* $\mathbf{F}_\omega$*-complete.*

Proof. It is sufficient to non-deterministically search for a satisfying model for $\varphi \in \mathcal{QTL}^{\#}_1$ of the prescribed size. However, since the class $\mathbf{F}_\omega$ is closed under exponentiation—and hence non-determinism—we have that the satisfiability problem for $\mathsf{FOLTL}^{\#}_{\mathsf{fin}}$ belongs to $\mathbf{F}_\omega$. The matching lower-bound is proved in [6], via a reduction from the reachability problem for *lossy counter machines* [10].□

3 Discussion

In [6], it was proved via reduction that the X-free fragment with sole quantifiers $\{\exists_{\leq 0}, \exists_{\leq 1}\}$ is decidable. However, this proof did not yield a effective upper-bound, as the decision procedure for the logic to which it is reduced depends upon Kruskal's Tree Theorem [4]. Indeed, in that same paper, the authors show that the logic in question is $\mathbf{F}_{\omega^\omega}$-hard, owing to a reduction from the reachability problem for *lossy channel systems* [2], making it strictly more complex than $\mathsf{FOLTL}^{\#}_{\mathsf{fin}}$, as demonstrated here. Note also that the choice of binary/unary encoding for the counting quantifiers does not have an effect on the complexity, with the limiting factor being the number of types for φ.

References

[1] Ackermann, W., *Zum hilbertschen aufbau der reellen zahlen*, Mathematische Annalen **99** (1928), pp. 118–133.

[2] Chambart, P. and P. Schnoebelen, *The ordinal recursive complexity of lossy channel systems*, in: *2008 23rd Annual IEEE Symposium on Logic in Computer Science*, IEEE, 2008, pp. 205–216.

[3] Figueira, D., S. Figueira, S. Schmitz and P. Schnoebelen, *Ackermannian and primitive-recursive bounds with Dickson's lemma*, in: *2011 IEEE 26th Annual Symposium on Logic in Computer Science*, IEEE, 2011, pp. 269–278.

[4] Gabelaia, D., A. Kurucz, F. Wolter and M. Zakharyaschev, *Non-primitive recursive decidability of products of modal logics with expanding domains*, Annals of Pure and Applied Logic **142** (2006), pp. 245 – 268.

[5] Hampson, C., *Decidable first-order modal logics with counting quantifiers.*, Advances in Modal Logic **11**, College Publications, 2016 pp. 382–400.

[6] Hampson, C. and A. Kurucz, *Undecidable propositional bimodal logics and one-variable first-order linear temporal logics with counting*, ACM Trans. Comput. Logic **16** (2015).

[7] Konev, B., F. Wolter and M. Zakharyaschev, *Temporal logics over transitive states*, in: R. Nieuwenhuis, editor, *Automated Deduction – CADE-20* (2005), pp. 182–203.

[8] Kurucz, A., F. Wolter, M. Zakharyaschev and D. M. Gabbay, *Many-dimensional modal logics: Theory and applications*, Elsevier, 2003.

[9] Schmitz, S., *Complexity hierarchies beyond elementary*, ACM Transactions on Computation Theory **8** (2016), pp. 1–36.

[10] Schnoebelen, P., *Revisiting Ackermann-hardness for lossy counter machines and reset Petri nets*, in: *International Symposium on Mathematical Foundations of Computer Science*, Springer, 2010, pp. 616–628.

Topological product of modal logics S4.1 and S4

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Abstract

We consider product of modal logics in topological semantics and prove that the topological product of S4.1 and S4 is the fusion of logics S4.1 and S4.

Keywords: Modal logic, topological semantics, product of modal logics, McKinsey axiom S4.1, S4

1 Introduction

The products of Kripke frames and there ware defined and studied by many authors (cf. [6,9,5]). It is a natural way to combine modal logics and to study the logics of two-dimensional structures. The same idea was used to define the product of topological spaces in [15]. Note in [15] that product of two topological spaces differ from the classical definition in topology. The main difference being that the result of the product from [15] is a set with two topologies: horizontal and vertical; and the classical product is a space with one topology called the product topology.

One of the main questions in this context is the following: given two complete modal logics L_1 and L_2, what is the logic of all possible products of corresponding structures (Kripke frames or topological spaces) with one being an L_1-structure and the second being an L_2-structure. It turns out that the result heavily depends on the type of structures. For example, the Kripke-frame-product $\mathsf{S4}$ times $\mathsf{S4}$ is the following logic (cf. [6])

$$\mathsf{S4} * \mathsf{S4} + \Box_1\Box_2 p \leftrightarrow \Box_2\Box_1 p + \Diamond_1\Box_2 p \to \Box_2\Diamond_1 p.\underline{(}cf)$$

where $\mathsf{S4} * \mathsf{S4}$ is the fusion of corresponding logics.

The topological product (precise definition is given in the next section) of $\mathsf{S4}$ times $\mathsf{S4}$ is just the fusion $\mathsf{S4} * \mathsf{S4}$ with no additional axioms (see [15])).

The notion of the product of topological spaces was generalized to the product of neighborhood frames in [11] and [7].

In this paper, we prove that the topological product $\mathsf{S4.1} \times_t \mathsf{S4} = \mathsf{S4.1} * \mathsf{S4}$.

2 Definitions and background

Let us establish the playground. Assume we have a countably infinite set of propositional letters PROP. A (modal) formula is defined recursively by using the Backus-Naur form as follows:

$$A ::= p \mid \bot \mid (A \to A) \mid \Box_i A,$$

where $p \in \text{PROP}$, and $\Box_i$ is a modal operator ($i = 1, \ldots, N$). Other connectives are introduced as abbreviations: classical connectives are expressed through $\bot$ and $\to$, and $\Diamond_i$ is a shortcut for $\neg\Box_i\neg$.

Definition 2.1 A *normal modal logic* (or *a logic*, for short) is a set of modal formulas closed under Substitution $\left(\frac{A(p)}{A(B)}\right)$, Modus Ponens $\left(\frac{A,\, A\to B}{B}\right)$ and Generalization rules $\left(\frac{A}{\Box_i A}\right)$, containing all the classical tautologies and the normality axioms:

$$\Box_i(p \to q) \to (\Box_i p \to \Box_i q).$$

$\mathsf{K_N}$ denotes *the minimal normal modal logic with n modalities* and $\mathsf{K} = \mathsf{K}_1$.

Let L be a logic and Γ be a set of formulas, then $\mathsf{L} + \Gamma$ denotes the minimal logic containing L and Γ. If $\Gamma = \{A\}$, then we write $\mathsf{L} + A$ rather than $\mathsf{L} + \{A\}$.

Logic $\mathsf{S4}$ is well known:

$$\mathsf{S4} = \mathsf{K} + \Box p \to p + \Box p \to \Box\Box p.$$

The notion of Kripke frames and truth conditions (the $\models$ relation) for them are well known. We refer the reader to [4]. The same goes for the topological spaces and topological models (see [1]).

For a class of topological spaces (Kripke frames) $\mathcal{C}$ the logic of it is

$$Log(\mathcal{C}) = \{A \mid \forall S \in \mathcal{C}(S \models A)\}$$

Note that if $\mathcal{C}$ is a class of birelational Kripke frames or bitopological spaces the logic you get will have two modalities.

Definition 2.2 Let $\mathfrak{X}_1 = (X_1, T_1)$ and $\mathfrak{X}_2 = (X_2, T_2)$ be two topological spaces. We define the *(bitopological) product* of them as the bitopological space $\mathfrak{X}_1 \times_t \mathfrak{X}_2 = (X_1 \times X_2, T_1^h, T_2^v)$. Topology T_1^h is the topology with the base $\{U \times \{x_2\} \mid U \in T_1 \;\&\; x_2 \in X_2\}$ and topology T_2^v is the topology with the base $\{\{x_1\} \times U \mid x_1 \in X_1 \;\&\; U \in T_2\}$.

Topological product of two topologically complete modal logics L_1 and L_2 is the following logic with two modalities:

$$L_1 \times_t L_2 = Log(\{\mathfrak{X}_1 \times_t \mathfrak{X}_2 \mid \mathfrak{X}_1, \mathfrak{X}_2 \text{ — topological spaces}, \mathfrak{X}_1 \models L_1, \mathfrak{X}_2 \models L_2\})$$

Theorem 2.3 ([15]) *The topological product of* $\mathsf{S4} \times \mathsf{S4}$ *is the fusion of* $\mathsf{S4}$ *with* $\mathsf{S4}$*. In symbols:*

$$\mathsf{S4} \times_t \mathsf{S4} = \mathsf{S4} * \mathsf{S4}.$$

Even more, the logic of product $\mathbb{Q} \times_t \mathbb{Q}$ *is also* $\mathsf{S4} * \mathsf{S4}$*.*

Ph. Kremer proved a surprising negative result that the logic of $\mathbb{R} \times_t \mathbb{R}$ is not $\mathsf{S4} * \mathsf{S4}$ (see [**?**]). This logic is still unknown.

There are also some results on the product of neighborhood frames closely related to the bitopological product (see [7,8,**?**,**?**]).

There were no completeness results of bitopological products for extensions of S4. In this paper we consider a well-known extension of S4 with formula $A1 = \Box\Diamond p \to \Diamond\Box p$. This formula is called the *McKinsey axiom* and it is well-studied both in the Kripke semantics and in topological semantics.

In the Kripke semantics this formula corresponds to the following property in the presence of S4: for an S4-frame $F = (W, R)$

$$F \models A1 \iff \forall w \in W \exists u \in W (wRu \wedge R(u) = \{u\}),$$

where $R(u) = \{t \mid uRt\}$. The proof is straightforward.

Let us recall some definitions from topology.

Definition 2.4 In topological space $\mathfrak{X}$ point x is *isolated* if set $\{x\}$ is open in $\mathfrak{X}$. $\mathfrak{X}$ is *weakly scattered* if the set of isolated points of $\mathfrak{X}$ is dense in $\mathfrak{X}$, that is if any open subset has an isolated point.

In topological semantics logic S4.1 was studied in [15,3,2]. It is known that S4.1 is the modal logic of the class of weakly scattered spaces. The proof can be found in [3].

3 Main result and further work

Theorem 3.1 $\mathsf{S4.1} \times_t \mathsf{S4} = \mathsf{S4.1} * \mathsf{S4}$.

For the further work we plan to investigate the following topics:

(i) Determine the logics $\mathsf{S4.1} \times_t \mathsf{S4.1}$, $\mathsf{S4.2} \times_t \mathsf{S4}$, $\mathsf{S4.2} \times_t \mathsf{S4.1}$ and $\mathsf{S4.2} \times_t \mathsf{S4.2}$. Hopefully, they will be equal to the fusions of the corresponding logics.

(ii) Add McKinsey axiom to transitive logics less then S4 like D4 and K4. From [**?**] and [7] we know that $\mathsf{D4} \times_t \mathsf{D4} = \mathsf{D4} * \mathsf{D4}$ and $\mathsf{K4} \times_t \mathsf{K4} = \mathsf{K4} * \mathsf{K4} + \Delta$, where Δ is the set of variable-free formulas of spatial form. We hope it will be possible to prove that adding axioms $A1$ and $A2$ will not add more axioms to the right-hand part of the equalities.

References

[1] Benthem, J., G. Bezhanishvili, B. Cate and D. Sarenac, *Multimodal logics of products of topologies*, Studia Logica **84** (2006), pp. 369–392.

[2] Bezhanishvili, G., D. Gabelaia and J. Lucero-Bryan, *Modal logics of metric spaces*, The Review of Symbolic Logic **8** (2015), pp. 178–191.

[3] Bezhanishvili, G. and J. Harding, *Modal logics of stone spaces*, Order **29** (2012), pp. 271–292.

[4] Blackburn, P., M. de Rijke and Y. Venema, "Modal Logic," Cambridge University Press, 2002.

[5] Gabbay, D., A. Kurucz and F. Wolter, "Many-dimensional modal logics : theory and applications," Studies in logic and the foundations of mathematics **148**, Elsevier, 2003.
[6] Gabbay, D. and V. Shehtman, *Products of modal logics. Part I*, Journal of the IGPL **6** (1998), pp. 73–146.
[7] Kudinov, A., *Modal logic of some products of neighborhood frames* (2012), pp. 386–394.
[8] Kudinov, A., *Neighbourhood frame product KxK.*, Advances in Modal Logic **10** (2014), pp. 373–386.
[9] Kurucz, A., *Combining modal logics*, Handbook of modal logic **3** (2007), pp. 869–924.
[10] Montague, R., *Universal grammar*, Theoria **36** (1970), pp. 373–398.
[11] Sano, K., *Axiomatizing hybrid products of monotone neighborhood frames*, Electr. Notes Theor. Comput. Sci. **273** (2011), pp. 51–67.
[12] Scott, D., *Advice on modal logic*, in: *Philosophical Problems in Logic: Some Recent Developments*, D. Reidel, 1970 pp. 143–173.
[13] Segerberg, K., *Two-dimensional modal logic*, Journal of Philosophical Logic **2** (1973), pp. 77–96.
[14] Shehtman, V., *Two-dimensional modal logic*, Mathematical Notices of USSR Academy of Science **23** (1978), pp. 417–424, (Translated from Russian).
[15] van Benthem, J. and G. Bezhanishvili, *Modal logics of space*, in: *Handbook of spatial logics*, Springer, 2007 pp. 217–298.

A Sketch of a Proof-Theoretic Semantics for Necessity

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Abstract

This paper considers proof-theoretic semantics for necessity within Dummett's and Prawitz's framework. Inspired by a system of Pfenning's and Davies's, the language of intuitionist logic is extended by a higher order operator which captures a notion of validity. A notion of relative necessary is defined in terms of it, which expresses a necessary connection between the assumptions and the conclusion of a deduction.

Keywords: proof-theoretic semantics, modal logic, necessity, higher-order rules.

1 Proof-Theoretic Semantics

Dummett and Prawitz do not consider how the meanings of modal operators may be given by their theory of meaning for the logical constants. To investigate in outline how this may be done is the purpose of this short paper.

According to proof-theoretic semantics, the rules governing a constant define its meaning. Prior's *tonk* shows that the rules cannot be arbitrary. Dummett and Prawitz impose the restriction that the introduction and elimination rules for a constant $*$ be *in harmony*, so that $*E$ does not license the deduction of more consequences from $A * B$ than are justified by the grounds for deriving it as specified by $*I$. (See [4], [11], [12], [13], [14].) A necessary condition for harmony is that deductions can be brought into normal form. A deduction is in *normal form* if it contains neither maximal formulas nor maximal segments. A *maximal formula* is one that is the conclusion of an I-rule and major premise of an E-rule. A *maximal segment* is a sequence of formulas all except the last of which are minor premises of $\vee E$ and the last one is major premise of an E-rule. [1]

[1] I am allowing myself a certain looseness in terminology, which, however, is quite common in the literature. Strictly speaking, Dummett distinguishes intrinsic harmony, stability and total harmony. Intrinsic harmony is captured by normalisation: the elimination rules of a constant are justified relative to the introduction rules. Stability is harmony together with a suitable converse: the introduction rules are also justified relative to the elimination rules. Total harmony obtains if the constant is conservative over the rest of the language. Dummett calls the permutative reduction steps to remove maximal segments 'auxiliary reduction steps'. Sometimes, as in the case of quantum disjunction, these cannot be carried out, which points to a defect in the rules for the connective from the meaning-theoretical perspective [4, 250,

Deductions in intuitionist logic **I** normalise [10, Ch 4]:

$$\vee I: \quad \frac{A}{A \vee B} \qquad \frac{B}{A \vee B} \qquad\qquad \vee E: \quad \frac{A \vee B \qquad \begin{array}{c}[A]^i \\ \Pi \\ C\end{array} \qquad \begin{array}{c}[B]^j \\ \Sigma \\ C\end{array}}{C}\; {\scriptstyle i,j}$$

$$\supset I: \quad \frac{\begin{array}{c}[A]^i \\ \Pi \\ B\end{array}}{A \supset B}\; {\scriptstyle i} \qquad \supset E: \quad \frac{A \supset B \qquad A}{B} \qquad\qquad \bot E: \quad \frac{\bot}{C}$$

$$\wedge I: \quad \frac{A \qquad B}{A \wedge B} \qquad\qquad \wedge E: \quad \frac{A \wedge B}{A} \qquad \frac{A \wedge B}{B}$$

The constants occur only in conclusions of *I*-rules and major premises of *E*-rules. Thus the conditions for using an *I*-rule and the consequences of using an *E*-rule are given independently of the constants.

The rules of **I** exemplify Dummett's notions of *full-bloodedness* and *molecularity* in the theory of meaning [5], [6]. A full-blooded theory of meaning characterises the knowledge of speakers in virtue of which they master a language in such a way that it exhibits how a speaker who does not yet understand an expression could acquire a grasp of it. A molecular theory of meaning does so piecemeal and specifies the meanings of the expressions of a language one group of expressions at a time. A speaker need not understand the constants of **I** in order to be informed about the conditions for the application of their *I*- and *E*-rules. To understand the grounds for deriving a formula with $*$ as main operator, or to understand the consequences that follow from it, a speaker only needs to grasp the meanings of some sentences, but not any sentences containing $*$. A speaker who does not already know the meanings of the constants of intuitionist logic could acquire a grasp of their meanings by learning the rules of inference of **I**. The rules are *informative*: the grounds and consequences of a formula with $*$ as main operator are given without reference to $*$. Its meaning is specified without presupposing that $*$ already has meaning.

Contrast the rules of **I** with standard rules for $\Box$ in **S4**:

$$\Box I: \quad \frac{A}{\Box A} \qquad\qquad \Box E: \quad \frac{\Box A}{A}$$

where in $\Box I$ all assumption on which A depends have the form $\Box B$. The conditions for applying $\Box I$ are not given independently of $\Box$. Thus they presuppose that $\Box$ is meaningful. Hence they do not define its meaning. Put in terms of

289]. Dummett observes that normalisation implies that each logical constant is conservative over the rest of the language [4, 250] and conjectures that 'intrinsic harmony implies total harmony in a context where stability prevails' [4, 290]. Dummett and Prawitz only count those segments as maximal that begin with the conclusion of an introduction rule. The more general notion used here is found in [15]. It is required for philosophical reasons. For more on these issues, see [8, Ch 2].

speakers' understanding, to be able to use $\Box I$ and to infer a formula of the form $\Box A$, a speaker already needs to know how to use formulas of the form $\Box B$ in deductions, and so the speaker already needs to know the meaning of $\Box B$. Thus a speaker could not acquire a grasp of the meaning of $\Box$ by being taught those rules. As a definition of the meaning of $\Box$, these rules are circular. The I-rule for $\Box$ presupposes that $\Box$ already has meaning. [2]

I propose that for the rules governing $*$ to define its meaning, they must satisfy a *Principle of Molecularity*: $*$ must not occur in the premises and discharged hypotheses of its I-rules, nor in any restrictions on their application, and $*$ must not occur in the minor premises and discharged hypotheses of its E-rules, nor in any restrictions on their application. Generalising, there should be no sequence of constants $*_1 ... *_n$ such that the rules for $*_i$ refer to $*_j$, $i < j$, and the rules for $*_n$ refer to $*_1$.

A promising system of modal logic from the present perspective was formalised by Pfenning and Davies [9]. It is based on Martin-Löf's account of judgements. They distinguish the judgment that a proposition is *true* from the judgement that a proposition is *valid*. $\vdash$ is interpreted as a hypothetical judgement. Validity is defined in terms of truth and hypothetical judgements, where $\cdot$ marks an empty collection of hypotheses and Γ are hypotheses of the form 'B *true*': (1) If $\cdot \vdash A\ true$, then $A\ valid$; (2) If $A\ valid$, then $\Gamma \vdash A\ true$.

Their system has axioms for the two kinds of hypotheses and rules for implication and necessity. Formulas assumed to be valid are to the left of the semi-colon, those assumed to be true to its right: [3]

$$\frac{}{\Delta; \Gamma, A\ true, \Gamma' \vdash A\ true}\ hyp \qquad \frac{}{\Delta, B\ valid, \Delta'; \Gamma \vdash B\ true}\ hyp*$$

$$\frac{\Delta; \Gamma, A\ true \vdash B\ true}{\Delta; \Gamma \vdash A \supset B\ true}\ {\supset}I \qquad \frac{\Delta; \Gamma \vdash A \supset B\ true \qquad \Delta; \Gamma \vdash A\ true}{\Delta; \Gamma \vdash B\ true}\ {\supset}E$$

$$\frac{\Delta; \cdot \vdash A\ true}{\Delta; \Gamma \vdash \Box A\ true}\ \Box I \qquad \frac{\Delta; \Gamma \vdash \Box A\ true \qquad \Delta, A\ valid; \Gamma \vdash C\ true}{\Delta; \Gamma \vdash C\ true}\ \Box E$$

Call this system **JM**. It is a fragment of intuitionist **S4**. A normalisation theorem can be proved for it. Its rules satisfy the Principle of Molecularity.

2 Modal Logic with Validity

In this section I reformulate, extend and generalise **JM**. The reformulation is three-fold. (1) I use a system of natural deduction not in sequent calculus style. (2) As any formula in **JM** is followed by either 'valid' or 'true', I drop the latter and simply write 'A'. This has a philosophical point: it accords with an account of logical inference as relating propositions, not judgements. (3) I

[2] Prawitz proves a normalisation theorem for intuitionist **S4** and **S5** [10, Ch 6]. Other such systems of intuitionist **S4** are formalised by Biermann and de Paiva [1] and von Plato [16]. Thus normalisation is not a sufficient condition for rules to define meaning.

[3] The restriction on Γ of clause (2) of the definition is not explained further. The point may well be to avoid circularity. It is effectively lifted in the axiom *hyp**.

do not treat validity as a judgement either, but as a sentential operator. The generalisation consists in the observation that validity is a relation between the assumptions and the conclusion of a deduction. The extension consists in formulating rules of inference for a higher level operator $\vdash$ for this generalised notion of validity. The rules for $\Box$ appeal to it. **HM** extends **I** by $\vdash$ and $\Box$.

Formulas of level 0 are those of **I** extended by $\Box$. Formulas of level 1 are all formulas $B_1...B_n \vdash A$, where $B_1...B_n, A$ are formulas of level 0, for $0 \leq n$. $B_1...B_n \vdash A$ can be derived if there is a deduction of A from $B_1...B_n$. Applying an elimination rule for $\vdash$, this is what we should get back. We may not know how A was derived from $B_1...B_n$, but as we know that there is such a deduction, the inference of A from $B_1...B_n$ is valid. $\vdash$ has the following rules:

$$VI: \frac{\begin{array}{c}[B_1]^{i_1}...[B_n]^{i_n}\\ \Pi \\ A\end{array}}{B_1...B_n \vdash A}\, i_1...i_n \qquad VE: \frac{\begin{array}{c}\Sigma\\ B_1...B_n \vdash A\end{array} \quad \begin{array}{c}\Xi_1\\ B_1\end{array} \quad ... \quad \begin{array}{c}\Xi_n\\ B_n\end{array}}{A}$$

where $B_1...B_n$, $0 \leq n$, are representatives of *all* the open assumption classes of Π in any order (as the Bs must be of level 0, there are no open assumptions of level 1). Vacuous discharge is allowed: a representative to the left of $\vdash$ may belong to an empty assumption class of Π; this corresponds to Thinning.

VI and VE are generalisations of Pfenning's and Davies's definition of validity cast into rules of a system of natural deduction. Next we generalise the I- and E-rules for necessity. $\Box$ is treated as a multi-grade constant which has one formula to its right and 0 to finite n formulas on its left. I abbreviate $B_1...B_n$ by Γ and write $\Gamma \vdash A$ instead of $B_1...B_n \vdash A$ and $[\Gamma]^{\overline{i}}$ instead of $[B_1]^{i_1}...[B_n]^{i_n}$. The rules for $\Box$ are:

$$\Box I: \frac{\begin{array}{c}[\Gamma]^{\overline{i}}\\ \Pi \\ A\end{array}}{\Gamma\Box A}\, \overline{i} \qquad \Box E: \frac{\begin{array}{c}\Sigma\\ \Gamma\Box A\end{array} \quad \begin{array}{c}[\Gamma \vdash A]^i\\ \Xi\\ C\end{array}}{C}\, i$$

where in $\Box I$, all open assumptions of level 0 of Π are in Γ (any other open assumptions are of level 1 and have the form $\Delta \vdash B$). Vacuous discharge is allowed. In $\Box E$, C is a 0-level formula. I propose to read $\Box$ as relative necessity. It expresses that there is a necessary connection between the formulas in Γ and A, or necessarily, A given Γ.[4] When Γ is empty, we get the usual unary necessity operator: it behaves as in intuitionist **S4**.

Maximal formulas of the form $\Gamma \vdash A$ are removed by the following reduction step:

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[4] For a few more thoughts on this modal notion, see [7]. It should be noted that on this reading, $\top$ is necessary relative to everything, while everything is necessary relative to $\bot$: the notion of relative necessity proposed here is not a relevant relative necessity.

$$\dfrac{\dfrac{\begin{matrix}[B_1]^{i_1}...[B_n]^{i_n}\\ \Pi \\ A\end{matrix}}{B_1...B_n \vdash A}{\scriptstyle i_1...i_n} \quad \begin{matrix}\Xi_1\\ B_1\end{matrix} \quad ... \quad \begin{matrix}\Xi_n\\ B_n\end{matrix}}{\begin{matrix}A\\ \Sigma\end{matrix}} \quad \rightsquigarrow \quad \begin{matrix}\Xi_1 & & \Xi_n\\ [B_1] & ... & [B_n]\\ & \Pi & \\ & A & \\ & \Sigma & \end{matrix}$$

The restrictions on VI and $\Box I$ require that all open formulas or all open 0-level formulas are discharged above their premises, and hence there can be no application of these rules in Π below $B_1...B_n$, except where an assumption class $[B]^i$ is empty. So the transformation cannot lead to any violations of restrictions on rules in Π. Any applications of those rules also remain correct in Σ, as the reduction procedure does not introduce new open assumptions into the deduction. For essentially the same reason, Prawitz's reduction procedures for maximal formulas and segments continue to work for the constants **HM** shares with **I**.

Removing maximal formulas $\Gamma\Box A$ is slightly more original:

$$\dfrac{\dfrac{\begin{matrix}[\Gamma]^{\overline{i}}\\ \Pi\\ A\end{matrix}}{\Gamma\Box A}{\scriptstyle i} \quad \begin{matrix}\dfrac{[\Gamma\vdash A]^j \quad \begin{matrix}\overline{\Sigma}\\ \Gamma\end{matrix}}{A}\\ \Xi\\ C\end{matrix}}{\begin{matrix}C\\ \Theta\end{matrix}}{\scriptstyle j} \quad \rightsquigarrow \quad \begin{matrix}\overline{\Sigma}\\ [\Gamma]\\ \Pi\\ A\\ \Xi\\ C\\ \Theta\end{matrix}$$

$\overline{\Sigma}$ are the deductions of the formulas in Γ. A maximal formula of type $\Gamma\Box A$ can only occur in the context on the left, unless $\Gamma \vdash A$ is discharged vacuously by $\Box E$, in which case its removal is trivial. The only thing one can do with $\Gamma \vdash A$ is to apply VE to it. Due to the restriction on C in $\Box E$ and the formation rules for the language of **HM**, such a formula cannot be assumed and immediately discharged by a rule. Due to the restrictions on VI and $\Box I$, there can be no applications of these rules below the Γs in Π (unless in the case of vacuous discharge, which is trivial): hence concluding the Γs with the deductions in $\overline{\Sigma}$ cannot lead to violations of rules in Π. Due to the restrictions on VI, there can be no application of that rule below A in Ξ, as there is at least the open assumption $\Gamma \vdash A$ that prevents such an application. If there is an application of $\Box I$ in Ξ, then all open assumptions of the deductions in $\overline{\Sigma}$ are of the form $\Delta \vdash B$, and hence they remain correct after the transformation. For similar reasons, applications of these rules in Θ also remain correct.

All reduction steps reduce the complexity of the deduction: a maximal segment is shortened, a maximal formula of higher degree than those that may be introduced by the reduction procedure removed. A standard induction over the complexity of deductions establishes the normalisation theorem for **HM**.

3 Conclusion

HM is a natural system of modal logic with higher order rules. It fulfils necessary conditions for a proof-theoretic account of the meaning of $\Box$. Deductions normalise. Its rules are harmonious and satisfy the molecularity principle. The meaning of $\Box$ is given in terms of the meaning of $\vdash$, the meaning of which is given in terms of inferences in **I**.

HM generalises **JM** in introducing a more general notion of validity and allowing validities to occur as conclusions of rules. But it remains close to **JM**, in that the restrictions on VI and the rules for $\Box$ are directly lifted from **JM**. A natural question is how the restrictions on VI could be loosened to allow further ways of deriving $\Gamma \vdash A$. The restriction on VI blocks a derivation of a version of cut: If (1) $\Gamma \vdash A$ and (2) $\Delta, A \vdash C$, then (3) $\Gamma, \Delta \vdash C$. It is possible to conclude A from (1) by assuming all formulas in Γ, and then to conclude C from (2) by assuming all formulas in Δ, but the restriction on VI prevents the conclusion of (3), as besides the 0-level formulas in Γ and Δ, the conclusion C depends on the undischarged first level formulas $\Gamma \vdash A$ and $\Delta, A \vdash C$.

Došen proposes systems of higher order sequents for intuitionist and classical **S4** and **S5** ([2], [3]), in which, he explains, $\Box A$ means 'A is assumed as a theorem'. This sounds similar to Pfenning's and Davies's account of modality. Došen's system implements a stricter distinction of levels of formulas and rules than **HM**. To the left and right of Došen's turnstile of level 2, there must be formulas of level 1, not of level 1 or 0. Thus transposed into a system of natural deduction, Došen's rules for $\Box$, which are of level 2, would require premises and conclusions of level 1. These rules are derivable using present the rules if VI may also be applied when all assumptions on which its premise depends are of level 1, i.e. of form $\Delta \vdash C$. Furthermore, with the restriction on VI so loosened that amongst the assumptions on which its premise depends there may be formulas of level 1, the version of cut mentioned in the previous paragraph becomes derivable. Modifying **HM** is an avenue for further research.

References

[1] Biermann, G. M. and V. C. V. de Paiva, *On an intuitionistic modal logic*, Studia Logica **65** (2000), pp. 383–416.

[2] Došen, K., *Sequent-systems for modal logic*, The Journal of Symbolic Logic **50** (1985), pp. 149–168.

[3] Došen, K., *Higher-level sequent systems for intuitionistic modal logic*, Publications der L'Institut Mathématique **38** (1986), pp. 3–12.

[4] Dummett, M., "The Logical Basis of Metaphysics," Cambridge, Mass.: Harvard University Press, 1993.

[5] Dummett, M., *What is a theory of meaning? (I)*, in: *The Seas of Language*, Oxford: Clarendon, 1993 pp. 1–33.

[6] Dummett, M., *What is a theory of meaning? (II)*, in: *The Seas of Language*, Oxford: Clarendon, 1993 pp. 34–93.

[7] Kürbis, N., *Proof-theoretic semantics, a problem with negation and prospects for modality*, The Journal of Philosophical Logic **44** (2015), pp. 713–727.

[8] Kürbis, N., "Proof and Falsity. A Logical Investigation," Cambridge University Press, 2019.

[9] Pfenning, F. and R. Davies, *A judgemental reconstruction of modal logic*, Mathematical Structures in Computer Science **11** (2001), pp. 511–540.

[10] Prawitz, D., "Natural Deduction," Stockholm, Göteborg, Uppsala: Almqvist and Wiksell, 1965.

[11] Prawitz, D., *Dummett on a theory of meaning and its impact on logic*, in: B. Taylor, editor, *Michael Dummett: Contributions to Philosophy*, Dordrecht: Nijhoff, 1987 pp. 117–165.

[12] Prawitz, D., *Meaning theory and anti-realism*, in: B. McGuiness, editor, *The Philosophy of Michael Dummett*, Dordrecht: Kluwer, 1994 pp. 79–89.

[13] Prawitz, D., *Meaning approached via proofs*, Synthese **148** (2006), pp. 507–524.

[14] Prawitz, D., *Pragmatist and verificationist theories of meaning*, in: R. Auxier and L. Hahn, editors, *The Philosophy of Michael Dummett*, Chicago: Open Court, 2007 pp. 455–481.

[15] Troestra, A. and H. Schwichtenberg, "Basic Proof Theory," Cambridge University Press, 2000, 2 edition.

[16] von Plato, J., *Normal derivability in modal logic*, Mathematical Logic Quarterly **51** (2005), pp. 632–638.

Fully structured proof theory for intuitionistic modal logics

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Abstract

We present a labelled sequent system and a nested sequent system for intuitionistic modal logics equipped with two relation symbols, one for the accessibility relation associated with the Kripke semantics for modal logics and one for the preorder relation associated with the Kripke semantics for intuitionistic logic. Both systems are in close correspondence with the bi-relational Kripke semantics for intuitionistic modal logic.

Keywords: Nested sequents, Labelled sequents, Intuitionistic modal logic, Proof theory.

1 Introduction

Structural proof theoretic accounts of modal logic can adopt the paradigm of *labelled deduction*, in the form of e.g. labelled sequent systems [12,7], or the one of *unlabelled deduction*, in the form of e.g. nested sequent systems [1,9].

These generalisations of the sequent framework, inspired by relational semantics, are needed to treat modalities uniformly. By extending the ordinary sequent structure with *one* extra element, either relational atoms between labels or nested bracketing, they encode respectively graphs or trees in the sequents, giving them enough power to represent modalities.

Similarly, proof systems have been designed for *intuitionistic* modal logic both as labelled [10] and as nested [11,4,3] sequent systems. Surprisingly, in nested and labelled sequents, extending the sequent structure with the same *one* extra element is enough to obtain sound and complete systems.

This no longer matches the relational semantics of these logics, which requires to combine *both* the relation for intuitionistic propositional logic and the one for modal logic. More importantly, it leads to deductive systems that are not entirely satisfactory; they cannot as modularly capture axiomatic extensions (or equivalently, restricted semantical conditions) and, in particular, can only provide decision procedures for a handful of them [10].

This lead us to develop a *fully structured* approach to intuitionistic modal proof theory capturing *both* the modal accessibility relation and the intuitionistic preorder relation. A *fully labelled* framework, described succintly in Section 3, has already allowed us to obtain modular systems for all intuitionistic Scott-Lemmon logics [6]. In an attempt to make this system amenable for proof-search and decision procedures, we have started investigated a *fully nested* framework, presented in Section 4. We would be particularly interested in a suitable system for logic $\mathsf{IS4}$, whose decidability is not known; we discuss this direction in Section 5.

2 Intuitionistic modal logic

The language of intuitionisitic modal logic is the one of intuitionistic propositional logic with the modal operators $\Box$ and $\Diamond$. Starting with a set $\mathcal{A}$ of atomic propositions, denoted a, modal formulas are constructed from the grammar:

$$A ::= a \mid \bot \mid (A \wedge A) \mid (A \vee A) \mid (A \supset A) \mid \Box A \mid \Diamond A$$

The axiomatisation of intuitionistic modal logic IK [8,2] is obtained from intuitionistic propositional logic by adding:

- the *necessitation rule*: $\Box A$ is a theorem if A is a theorem; and
- the following five variants of the *distributivity axiom*:

$$\begin{array}{lll} \mathsf{k}_1\colon\ \Box(A \supset B) \supset (\Box A \supset \Box B) & \mathsf{k}_3\colon\ \Diamond(A \vee B) \supset (\Diamond A \vee \Diamond B) & \mathsf{k}_5\colon\ \Diamond\bot \supset \bot \\ \mathsf{k}_2\colon\ \Box(A \supset B) \supset (\Diamond A \supset \Diamond B) & \mathsf{k}_4\colon\ (\Diamond A \supset \Box B) \supset \Box(A \supset B) & \end{array}$$

Definition 2.1 A *bi-relational frame* consists of a set of worlds W equipped with an accessibility relation R and a preorder $\leq$ satisfying:

(F_1) For $x, y, z \in W$, if xRy and $y \leq z$, there exists u s.t. $x \leq u$ and uRz.

(F_2) For $x, y, z \in W$, if $x \leq y$ and xRz, there exists u s.t. yRu and $z \leq u$.

Definition 2.2 A *bi-relational model* is a bi-relational frame with a monotone valuation function $V\colon W \to 2^{\mathcal{A}}$.

We write $x \Vdash a$ if $a \in V(x)$ and, by definition, it is never the case that $x \Vdash \bot$. The relation $\Vdash$ is extended to all formulas by induction, following the rules for both intuitionistic and modal Kripke models:

$$\begin{array}{lll} x \Vdash A \wedge B & \text{iff} & x \Vdash A \text{ and } x \Vdash B \\ x \Vdash A \vee B & \text{iff} & x \Vdash A \text{ or } x \Vdash B \\ x \Vdash A \supset B & \text{iff} & \text{for all } y \text{ with } x \leq y, \text{ if } y \Vdash A \text{ then } y \Vdash B \\ x \Vdash \Box A & \text{iff} & \text{for all } y \text{ and } z \text{ with } x \leq y \text{ and } yRz, z \Vdash A \\ x \Vdash \Diamond A & \text{iff} & \text{there exists a } y \text{ such that } xRy \text{ and } y \Vdash A \end{array} \qquad (1)$$

Definition 2.3 A formula A is *valid* in a frame $\langle W, R, \leq \rangle$, if for all monotone valuations V and for all $w \in W$, we have $w \Vdash A$

Theorem 2.4 ([2,8]) *A formula A is a theorem of IK if and only if A is valid in every bi-relational frame.*

$$\mathsf{id}\ \frac{}{\mathcal{B}, x \leq y, \mathcal{L}, x{:}A \Rightarrow \mathcal{R}, y{:}A} \qquad \bot_{\mathsf{L}}\ \frac{}{\mathcal{B}, \mathcal{L}, x{:}\bot \Rightarrow \mathcal{R}}$$

$$\wedge_{\mathsf{L}}\ \frac{\mathcal{B}, \mathcal{L}, x{:}A, x{:}B \Rightarrow \mathcal{R}}{\mathcal{B}, \mathcal{L}, x{:}A \wedge B \Rightarrow \mathcal{R}} \qquad \wedge_{\mathsf{R}}\ \frac{\mathcal{B}, \mathcal{L} \Rightarrow \mathcal{R}, x{:}A \quad \mathcal{B}, \mathcal{L} \Rightarrow \mathcal{R}, x{:}B}{\mathcal{B}, \mathcal{L} \Rightarrow \mathcal{R}, x{:}A \wedge B}$$

$$\vee_{\mathsf{L}}\ \frac{\mathcal{B}, \mathcal{L}, x{:}A \Rightarrow \mathcal{R} \quad \mathcal{B}, \mathcal{L}, x{:}B \Rightarrow \mathcal{R}}{\mathcal{B}, \mathcal{L}, x{:}A \vee B \Rightarrow \mathcal{R}} \qquad \vee_{\mathsf{R}}\ \frac{\mathcal{B}, \mathcal{L} \Rightarrow \mathcal{R}, x{:}A, x{:}B}{\mathcal{B}, \mathcal{L} \Rightarrow \mathcal{R}, x{:}A \vee B}$$

$$\supset_{\mathsf{L}}\ \frac{\mathcal{B}, \mathcal{L}, x \leq y, y{:}A \Rightarrow \mathcal{R}, y{:}B}{\mathcal{B}, \mathcal{L} \Rightarrow \mathcal{R}, x{:}A \supset B}\ y \text{ fresh}$$

$$\supset_{\mathsf{R}}\ \frac{\mathcal{B}, x \leq y, \mathcal{L} \Rightarrow \mathcal{R}, y{:}A \quad \mathcal{B}, x \leq y, \mathcal{L}, y{:}B \Rightarrow \mathcal{R}}{\mathcal{B}, x \leq y, \mathcal{L}, x{:}A \supset B \Rightarrow \mathcal{R}}$$

$$\Box_{\mathsf{L}}\ \frac{\mathcal{B}, x \leq y, yRz, \mathcal{L}, x{:}\Box A, z{:}A \Rightarrow \mathcal{R}}{\mathcal{B}, x \leq y, yRz, \mathcal{L}, x{:}\Box A \Rightarrow \mathcal{R}} \qquad \Box_{\mathsf{R}}\ \frac{\mathcal{B}, x \leq y, yRz, \mathcal{L} \Rightarrow \mathcal{R}, z{:}A}{\mathcal{B}, \mathcal{L} \Rightarrow \mathcal{R}, x{:}\Box A}\ y, z \text{ fresh}$$

$$\Diamond_{\mathsf{L}}\ \frac{\mathcal{B}, xRy, \mathcal{L}, y{:}A \Rightarrow \mathcal{R}}{\mathcal{B}, \mathcal{L}, x{:}\Diamond A \Rightarrow \mathcal{R}}\ y \text{ fresh} \qquad \Diamond_{\mathsf{R}}\ \frac{\mathcal{B}, xRy, \mathcal{L} \Rightarrow \mathcal{R}, x{:}\Diamond A, y{:}A}{\mathcal{B}, xRy, \mathcal{L} \Rightarrow \mathcal{R}, x{:}\Diamond A}$$

....

$$\mathsf{refl}_{\leq}\ \frac{\mathcal{B}, x \leq x, \mathcal{L} \Rightarrow \mathcal{R}}{\mathcal{B}, \mathcal{L} \Rightarrow \mathcal{R}} \qquad \mathsf{trans}_{\leq}\ \frac{\mathcal{B}, x \leq y, y \leq z, x \leq z, \mathcal{L} \Rightarrow \mathcal{R}}{\mathcal{B}, x \leq y, y \leq z, \mathcal{L} \Rightarrow \mathcal{R}}$$

$$\mathsf{F}_1\ \frac{\mathcal{B}, xRy, y \leq z, x \leq u, uRz, \mathcal{L} \Rightarrow \mathcal{R}}{\mathcal{B}, xRy, y \leq z, \mathcal{L} \Rightarrow \mathcal{R}}\ u \text{ fresh}$$

$$\mathsf{F}_2\ \frac{\mathcal{B}, xRy, x \leq z, y \leq u, zRu, \mathcal{L} \Rightarrow \mathcal{R}}{\mathcal{B}, xRy, x \leq z, \mathcal{L} \Rightarrow \mathcal{R}}\ u \text{ fresh}$$

Fig. 1. System $\mathsf{labIK}_{\leq}$

3 Fully labelled sequent calculus

Echoing the definition of bi-relational structures, we consider an extension of labelled deduction to the intuitionistic setting that uses two sorts of relational atoms, one for the modal accessibility relation R and another one for the intuitionistic preorder relation $\leq$ (similarly to [5] for epistemic logic).

Definition 3.1 A two-sided intuitionistic *fully labelled sequent* is of the form $\mathcal{B}, \mathcal{L} \Rightarrow \mathcal{R}$ where $\mathcal{B}$ denotes a set of relational atoms xRy and preorder atoms $x \leq y$, and $\mathcal{L}$ and $\mathcal{R}$ are multi-sets of labelled formulas $x{:}A$ (for x and y taken from a countable set of labels and A an intuitionistic modal formula).

We obtain a proof system $\mathsf{labIK}_{\leq}$, displayed on Figure 1, for intuitionistic modal logic in this formalism. Most rules are similar to the ones of Simpson [10], but some are more explicitly in correspondence with the semantics by using the preorder atoms. For instance, the rules introducing the $\Box$-connective correspond to (1). Furthermore, our system must incorporate the conditions (F_1) and (F_2) into the deductive rules F_1 and F_2, and rules $\mathsf{refl}_{\leq}$ and $\mathsf{trans}_{\leq}$ are necessary to ensure that the preorder atoms behave as a preorder on labels.

$$\mathsf{id}\,\frac{}{\Gamma\{A^\bullet, A^\circ\}} \qquad \bot_{\mathsf{L}}\,\frac{}{\Gamma\{\bot^\bullet\}}$$

$$\wedge_{\mathsf{L}}\,\frac{\Gamma\{A^\bullet, B^\bullet\}}{\Gamma\{A \wedge B^\bullet\}} \quad \wedge_{\mathsf{R}}\,\frac{\Gamma\{A^\circ\} \quad \Gamma\{B^\circ\}}{\Gamma\{A \wedge B^\circ\}} \quad \vee_{\mathsf{L}}\,\frac{\Gamma\{A^\bullet\} \quad \Gamma\{B^\bullet\}}{\Gamma\{A \vee B^\bullet\}} \quad \vee_{\mathsf{R}}\,\frac{\Gamma\{A^\circ, B^\circ\}}{\Gamma\{A \vee B^\circ\}}$$

$$\supset_{\mathsf{L}}\,\frac{\Gamma_1\{A \supset B^\bullet, A^\circ\} \quad \Gamma_1\{B^\bullet\}}{\Gamma_1\{A \supset B^\bullet\}} \qquad \supset_{\mathsf{R}}\,\frac{\Gamma\{[\![A^\bullet, B^\circ]\!]\}}{\Gamma\{A \supset B^\circ\}}$$

$$\Box_{\mathsf{L}}\,\frac{\Gamma_1\{\Box A^\bullet, [A^\bullet, \Gamma_2]\}}{\Gamma_1\{\Box A^\bullet, [\Gamma_2]\}} \quad \Box_{\mathsf{R}}\,\frac{\Gamma\{[\![[A^\circ]]\!]\}}{\Gamma\{\Box A^\circ\}} \quad \Diamond_{\mathsf{L}}\,\frac{\Gamma\{[A^\bullet]\}}{\Gamma\{\Diamond A^\bullet\}} \quad \Diamond_{\mathsf{R}}\,\frac{\Gamma_1\{\Diamond A^\circ, [A^\circ, \Gamma_2]\}}{\Gamma_1\{\Diamond A^\circ, [\Gamma_2]\}}$$

$$\mathsf{mon_R}\,\frac{\Gamma_1\{A^\circ, [\![A^\circ, \Gamma_2]\!]\}}{\Gamma_1\{[\![A^\circ, \Gamma_2]\!]\}} \quad \mathsf{mon_L}\,\frac{\Gamma_1\{A^\bullet, [\![A^\bullet, \Gamma_2]\!]\}}{\Gamma_1\{A^\bullet, [\![\Gamma_2]\!]\}} \quad \mathsf{F_1}\,\frac{\Gamma_1\{[\Gamma_2], [\![[\Gamma_3]]\!]\}}{\Gamma_1\{[\Gamma_2, [\![\Gamma_3]\!]]\}}$$

Fig. 2. System $\mathsf{nIK}_{\leq}$

Theorem 3.2 *For any formula A, the following are equivalent.*

(i) *A is a theorem of* IK

(ii) *A is provable in* $\mathsf{labIK}_{\leq} + \mathsf{cut}$ *with* $\quad \mathsf{cut}\,\frac{\mathcal{B}_1, \mathcal{L} \Rightarrow \mathcal{R}, z{:}C \quad \mathcal{B}_2, \mathcal{L}, z{:}C \Rightarrow \mathcal{R}}{\mathcal{B}_1, \mathcal{B}_2, \mathcal{L} \Rightarrow \mathcal{R}}$

(iii) *A is provable in* $\mathsf{labIK}_{\leq}$

The proof is a careful adaptation of standard techniques (see [6] for details).

4 Fully nested sequent calculus

In standard nested sequent notation, brackets $[\cdot]$ are used to indicate the parent-child relation in the modal accessibility tree. $(\cdot)^\bullet$ and $(\cdot)^\circ$ annotations are used to indicate that the formulas would occur on the left-hand-side or right-hand-side of a sequent, respectively, in the absence of the sequent arrow.

To make it fully structured again, we enhance the structure with a second type of bracketting $[\![\cdot]\!]$ to encode the preorder relation.

Definition 4.1 A two-sided intuitionistic *fully nested sequent* is constructed from the grammar: $\Gamma ::= \varnothing \mid A^\bullet, \Gamma \mid A^\circ, \Gamma \mid [\Gamma] \mid [\![\Gamma]\!]$

The obtained nested sequent calculus $\mathsf{nIK}_{\leq}$ is displayed in Figure 2. The idea is similar to the fully labelled calculus but the shift of paradigm allowed us to make different design choices. In particular, the underlying tree-structure prevents us to express the rule F_2, but its absence is offset by the *monotonicity rules* $\mathsf{mon_L}$ and $\mathsf{mon_R}$, which were admissible in $\mathsf{labIK}_{\leq}$. Another benefit of this addition is that rules $\mathsf{refl}_{\leq}$ and $\mathsf{trans}_{\leq}$ do not need any equivalent here.

5 Extensions: example of transitivity

As mentioned in the introduction, one of our motivation is to investigate decision procedure for axiomatic extensions of IK, for instance $\mathsf{IS4}$, intuitionistic logic of reflexive transitive frames. We will therefore illustrate our approach taking transitivity as a test-case.

The frame condition of transitivity ($\forall xyz.\, xRy \wedge yRz \supset xRz$) can be axiomatised by adding to IK the conjunction of the two versions of the $\mathsf{4}$-axiom:

$$\mathsf{4}_\Box : \Box A \supset \Box\Box A \qquad\qquad \mathsf{4}_\Diamond : \Diamond\Diamond A \supset \Diamond A$$

which are equivalent in classical modal logic. However, in intuitionistic modal logic they are not and they can be added to IK independently. From [8] we know they are in correspondence respectively with the following frame conditions:

$$\forall xyz.((xRy \wedge yRz) \supset \exists u.(x \leq u \wedge uRz)) \quad \forall xyz.((xRy \wedge yRz) \supset \exists u.(z \leq u \wedge xRu)) \quad (2)$$

Following Simpson [10] we could extend our basic sequent system for IK to $\mathsf{IK4} = \mathsf{IK} + (\mathsf{4}_\Box \wedge \mathsf{4}_\Diamond)$ with the rule

$$\mathsf{trans}_R \frac{\mathcal{B}, wRv, vRu, wRu, \mathcal{L} \Rightarrow \mathcal{R}}{\mathcal{B}, wRv, vRu, \mathcal{L} \Rightarrow \mathcal{R}}$$

Incorporating the preorder symbol into the syntax too, allowed us however to translate the conditions in (2) into separate inference rules for $\mathsf{4}_\Box$ and $\mathsf{4}_\Diamond$:

$$\mathsf{4}_\Box \frac{\mathcal{B}, xRy, yRz, uRz, x \leq u, \mathcal{L} \Rightarrow \mathcal{R}}{\mathcal{B}, xRy, yRz, \mathcal{L} \Rightarrow \mathcal{R}}\ u \text{ fresh} \quad \mathsf{4}_\Diamond \frac{\mathcal{B}, xRy, yRz, xRu, z \leq u, \mathcal{L} \Rightarrow \mathcal{R}}{\mathcal{B}, xRy, yRz, \mathcal{L} \Rightarrow \mathcal{R}}\ u \text{ fresh}$$

These extensions for $\mathsf{labIK}_\leq$ are sound and complete; more generally, Theorem 3.2 can be extended to the class of intuitionisitc Scott-Lemmon logics [6].

Similar results for the fully nested sequent system are subject of ongoing study. Previous nested systems for intuitionistic modal logics [11,4] can be extended from IK to $\mathsf{IK4}$ by simply adding the following rules:

$$\Box_{\mathsf{L4}} \frac{\Gamma_1\{\Box A^\bullet, [\Box A^\bullet, \Gamma_2]\}}{\Gamma_1\{\Box A^\bullet, [\Gamma_2]\}} \quad \Diamond_{\mathsf{R4}} \frac{\Gamma_1\{\Diamond A^\circ, [\Diamond A^\circ, \Gamma_2]\}}{\Gamma_1\{\Diamond A^\circ, [\Gamma_2]\}}$$

These rules are logical rather than structural as their labelled counterpart, making them usually more suitable for proof search procedures.

References

[1] Brünnler, K., *Deep sequent systems for modal logic*, Archive for Mathematical Logic **48** (2009), pp. 551–577.

[2] Fischer Servi, G., *Axiomatizations for some intuitionistic modal logics*, Rend. Sem. Mat. Univers. Politecn. Torino **42** (1984), pp. 179–194.

[3] Galmiche, D. and Salhi, Y., *Tree-sequent calculi and decision procedures for intuitionistic modal logics*, Journal of Logic and Computation **28** (2018), pp. 967–989.

[4] Kuznets, R. and Straßburger, L., *Maehara-style modal nested calculi*, Archive for Mathematical Logic **58** (2018), pp. 359–385.

[5] Maffezioli, P. and Naibo, A. and Negri, S.,*The Church–Fitch knowability paradox in the light of structural proof theory*, Synthese **190** (2013), pp. 2677–2716.

[6] Marin, S., Morales, M. and Straßburger, L., *A fully labelled proof system for intuitionistic modal logics* (2019), ArXiv preprint.

[7] Negri, S., *Proof analysis in modal logic*, J. of Philosophical Logic **34** (2005), pp. 107–128.

[8] Plotkin, G. and Stirling, C., *A framework for intuitionistic modal logics*, Proceedings of the 1st Conf. on Theoretical Aspects of Reasoning about Knowledge (1986), pp. 399–406.

[9] Poggiolesi, F., *The method of tree-hypersequents for modal propositional logic*, Towards mathematical philosophy (2009), pp. 31–51.

[10] Simpson, A. K., *The proof theory and semantics of intuitionistic modal logic*, Ph.D. thesis, University of Edinburgh. College of Science and Engineering (1994).

[11] Straßburger, L., *Cut Elimination in Nested Sequents for Intuitionistic Modal Logics*, FoSSaCS'13,LNCS **7794** (2013), pp. 209–224.

[12] Viganò, L., *Labelled non-classical logics*, Kluwer Academic Publisher (2000).

Ecumenical modal logic (short version)

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Abstract

The discussion about how to put together Gentzen's systems for classical and intuitionistic logic in a single unified system is back in fashion. Indeed, recently Prawitz and others have been discussing the so called Ecumenical Systems, where connectives from these logics can co-exist in peace. In Prawitz' system, the classical logician and the intuitionistic logician would share the universal quantifier, conjunction, negation, and the constant for the absurd, but they would each have their own existential quantifier, disjunction, and implication, with different meanings. Prawitz' main idea is that these different meanings are given by a semantical framework that can be accepted by both parties. In a recent work, Ecumenical sequent calculi and a nested system were presented, and some very interesting proof theoretical properties of the systems were established. In this work we extend Prawitz' Ecumenical idea to alethic K-modalities.

Keywords: Ecumenical systems, modalities, labeled systems, Kripke semantics.

1 Introduction

In [3] Dag Prawitz proposed a natural deduction system for what was later called *Ecumenical logic* (EL), where classical and intuitionistic logic could coexist in peace. In this system, the classical logician and the intuitionistic logician would share the universal quantifier, conjunction, negation, and the constant for the absurd (*the neutral connectives*), but they would each have their own existential quantifier, disjunction, and implication, with different meanings. Prawitz' main idea is that these different meanings are given by a semantical framework that can be accepted by both parties.

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INITIAL AND STRUCTURAL RULES

$$\frac{}{A,\Gamma \Rightarrow A}\ \mathsf{init} \qquad \frac{\Gamma \Rightarrow \bot}{\Gamma \Rightarrow A}\ \mathsf{W}$$

PROPOSITIONAL RULES

$$\frac{A,B,\Gamma \Rightarrow C}{A \wedge B,\Gamma \Rightarrow C}\ \wedge L \quad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\ \wedge R \quad \frac{A,\Gamma \Rightarrow C \quad B,\Gamma \Rightarrow C}{A \vee_i B,\Gamma \Rightarrow C}\ \vee_i L \quad \frac{\Gamma \Rightarrow A_j}{\Gamma \Rightarrow A_1 \vee_i A_2}\ \vee_i R_j$$

$$\frac{A,\Gamma \Rightarrow \bot \quad B,\Gamma \Rightarrow \bot}{A \vee_c B,\Gamma \Rightarrow \bot}\ \vee_c L \quad \frac{\Gamma,\neg A,\neg B \Rightarrow \bot}{\Gamma \Rightarrow A \vee_c B}\ \vee_c R \quad \frac{A \rightarrow_i B,\Gamma \Rightarrow A \quad B,\Gamma \Rightarrow C}{\Gamma, A \rightarrow_i B \Rightarrow C}\ \rightarrow_i L$$

$$\frac{\Gamma,A \Rightarrow B}{\Gamma \Rightarrow A \rightarrow_i B}\ \rightarrow_i R \quad \frac{A \rightarrow_c B,\Gamma \Rightarrow A \quad B,\Gamma \Rightarrow \bot}{A \rightarrow_c B,\Gamma \Rightarrow \bot}\ \rightarrow_c L \quad \frac{\Gamma,A,\neg B \Rightarrow \bot}{\Gamma \Rightarrow A \rightarrow_c B}\ \rightarrow_c R$$

$$\frac{\neg A,\Gamma \Rightarrow A}{\neg A,\Gamma \Rightarrow \bot}\ \neg L \quad \frac{\Gamma,A \Rightarrow \bot}{\Gamma \Rightarrow \neg A}\ \neg R \quad \frac{}{\bot,\Gamma \Rightarrow A}\ \bot L \quad \frac{P_i,\Gamma \Rightarrow \bot}{P_c,\Gamma \Rightarrow \bot}\ L_c \quad \frac{\Gamma,\neg P_i \Rightarrow \bot}{\Gamma \Rightarrow P_c}\ R_c$$

QUANTIFIERS

$$\frac{A[y/x],\forall x.A,\Gamma \Rightarrow C}{\forall x.A,\Gamma \Rightarrow C}\ \forall L \quad \frac{\Gamma \Rightarrow A[y/x]}{\Gamma \Rightarrow \forall x.A}\ \forall R$$

$$\frac{A[y/x],\Gamma \Rightarrow C}{\exists_i x.A,\Gamma \Rightarrow C}\ \exists_i L \quad \frac{\Gamma \Rightarrow A[y/x]}{\Gamma \Rightarrow \exists_i x.A}\ \exists_i R \quad \frac{A[y/x],\Gamma \Rightarrow \bot}{\exists_c x.A,\Gamma \Rightarrow \bot}\ \exists_c L \quad \frac{\Gamma,\forall x.\neg A \Rightarrow \bot}{\Gamma \Rightarrow \exists_c x.A}\ \exists_c R$$

Fig. 1. Ecumenical sequent system LEci. In rules $\forall R, \exists_i L, \exists_c L$, the eigenvariable y is fresh.

While proof-theoretical aspects were also considered, his work was more focused on investigating the philosophical significance of the fact that classical logic can be translated into intuitionistic logic.

In this work, we propose an extension of EL with the alethic modalities of *necessity* and *possibility*. There are many choices to be made and many relevant questions to be asked, *e.g.*: what is the ecumenical interpretation of Ecumenical modalities? Should we add classical, intuitionistic, or neutral versions for modal connectives? What is really behind the difference between the classical and intuitionistic notions of truth?

We propose an answer for these questions in the light of Simpson's meta-logical interpretation of modalities [4] by embedding the expected semantical behavior of the modal operator into the Ecumenical first-order logic.

2 The system LEci

The language $\mathcal{L}$ used for Ecumenical systems is described as follows. We will use a subscript c for the classical meaning and i for the intuitionistic, dropping such subscripts when formulae/connectives can have either meaning.

Classical and intuitionistic n-ary predicate symbols ($P_c, P_i, \ldots$) co-exist in $\mathcal{L}$ but have different meanings. The neutral logical connectives $\{\bot, \neg, \wedge, \forall\}$ are common for classical and intuitionistic fragments, while $\{\rightarrow_i, \vee_i, \exists_i\}$ and $\{\rightarrow_c, \vee_c, \exists_c\}$ are restricted to intuitionistic and classical interpretations, respectively.

The sequent system LEci (Fig. 1) was presented in [2] as the sequent counterpart of Prawitz natural deduction system. Observe that the rules R_c and L_c describe the intended meaning of a classical predicate P_c from an intuitionistic predicate P_i.

The following are easily provable in LEci:

(i) $\vdash_{\mathsf{LEci}} (A \vee_c B) \leftrightarrow_i \neg(\neg A \wedge \neg B)$;

(ii) $\vdash_{\mathsf{LEci}} (A \rightarrow_c B) \leftrightarrow_i \neg(A \wedge \neg B)$;

(iii) $\vdash_{\mathsf{LEci}} (\exists_c x.A) \leftrightarrow_i \neg(\forall x.\neg A)$.

(iv) $\vdash_{\mathsf{LEci}} \forall x.A \rightarrow_i \neg\exists_c x.\neg A$ but $\nvdash_{\mathsf{LEci}} \neg\exists_c x.\neg A \rightarrow_i \forall x.A$ in general.

Theorems (i) to (iii) are of interest since they relate the classical and the neutral operators: the classical connectives can be defined using negation, conjunction, and the universal quantifier. Observe that (iii) and (iv) reveal the asymmetry between definability of quantifiers: while the classical existential can be defined from the universal quantification, the other way around is not true, in general.

3 Ecumenical modalities

The language of *(propositional, normal) modal formulas* consists of a denumerable set $\mathcal{P}$ of propositional symbols and a set of propositional connectives enhanced with the unary *modal operators* $\Box$ and $\Diamond$ concerning necessity and possibility, respectively [1].

We will follow the approach in [4], where a modal logic is characterized by the respective interpretation of the modal model in the meta-theory (called *meta-logical characterization*).

Formally, given a variable x, we recall the standard translation $[\cdot]_x$ from modal formulas into first-order formulas with at most one free variable x: for any $P \in \mathcal{P}$, a unary predicate symbol P is associated to it and $[P]_x := P(x)$; $[\bot]_x := \bot$; for any binary connective $\star$, $[A \star B]_x := [A]_x \star [B]_x$; for the modal connectives

$$[\Box A]_x \quad := \quad \forall y(R(x,y) \rightarrow [A]_y) \qquad [\Diamond A]_x \quad := \quad \exists y(R(x,y) \wedge [A]_y)$$

where $R(x,y)$ is a binary predicate.

The *object* modal logic ML is then interpreted in the first-order *meta* logic FOL as

$$\vdash_{ML} A \quad \text{iff} \quad \vdash_{FOL} \forall x.[A]_x$$

Hence, if FOL is classical, the former definition characterizes the classical modal logic K [1], while if it is intuitionistic, it characterizes the intuitionistic modal logic IK [4].

In this work, we will adopt first-order EL as the meta-theory (given by the system LEci), hence characterizing what we will defined as the ecumenical modal logic EK.

3.1 An Ecumenical view of modalities

The language of *Ecumenical modal formulas* consists of a denumerable set $\mathcal{P}$ of (Ecumenical) propositional symbols and the set of Ecumenical connectives enhanced with unary *Ecumenical modal operators*. There is no canonical definition of constructive or intuitionistic modal logics. Here we will mostly follow the approach in [4] for justifying our choices for the Ecumenical interpretation for *possibility* and *necessity*.

The ecumenical translation $[\cdot]^e_x$ from propositional ecumenical formulas into LEci is defined in the same way as the modal translation $[\cdot]_x$ in the last section. For the case of modal connectives, our proposal is that the box modality is a *neutral connective*, while the diamond has two possible interpretations: classical and intuitionistic, as its leading connective is an existential quantifier. Hence we should consider the ecumenical modalities: $\Box, \Diamond_i, \Diamond_c$, determined by the translations

INITIAL AND STRUCTURAL RULES

$$\frac{}{x : A, \Gamma \vdash x : A}\ \mathsf{init} \qquad \frac{\Gamma \vdash y : \bot}{\Gamma \vdash x : A}\ \mathsf{W}$$

PROPOSITIONAL RULES

$$\frac{x : A, x : B, \Gamma \vdash z : C}{x : A \wedge B, \Gamma \vdash z : C}\ \wedge L \qquad \frac{\Gamma \vdash x : A \quad \Gamma \vdash x : B}{\Gamma \vdash x : A \wedge B}\ \wedge R \qquad \frac{x : A, \Gamma \vdash z : C \quad x : B, \Gamma \vdash z : C}{x : A \vee_i B, \Gamma \vdash z : C}\ \vee_i L$$

$$\frac{\Gamma \vdash x : A_j}{\Gamma \vdash x : A_1 \vee_i A_2}\ \vee_i R_j \qquad \frac{x : A, \Gamma \vdash x : \bot \quad x : B, \Gamma \vdash x : \bot}{x : A \vee_c B, \Gamma \vdash x : \bot}\ \vee_c L \qquad \frac{\Gamma, x : \neg A, x : \neg B \vdash x : \bot}{\Gamma \vdash x : A \vee_c B}\ \vee_c R$$

$$\frac{x : A \rightarrow_i B, \Gamma \vdash x : A \quad x : B, \Gamma \vdash z : C}{x : A \rightarrow_i B, \Gamma \vdash z : C}\ \rightarrow_i L \qquad \frac{x : A, \Gamma \vdash x : B}{\Gamma \vdash x : A \rightarrow_i B}\ \rightarrow_i R \qquad \frac{x : A, \Gamma \vdash x : \bot}{\Gamma \vdash x : \neg A}\ \neg R$$

$$\frac{x : A \rightarrow_c B, \Gamma \vdash x : A \quad x : B, \Gamma \vdash x : \bot}{x : A \rightarrow_c B, \Gamma \vdash x : \bot}\ \rightarrow_c L \qquad \frac{x : A, x : \neg B, \Gamma \vdash x : \bot}{\Gamma \vdash x : A \rightarrow_c B}\ \rightarrow_c R \qquad \frac{x : \neg A, \Gamma \vdash x : A}{x : \neg A, \Gamma \vdash x : \bot}\ \neg L$$

$$\frac{}{x : \bot, \Gamma \vdash z : C}\ \bot \qquad \frac{\Gamma, x : P_i \vdash x : \bot}{\Gamma, x : P_c \vdash x : \bot}\ L_c \qquad \frac{\Gamma, x : \neg P_i \vdash x : \bot}{\Gamma \vdash x : P_c}\ R_c$$

MODAL RULES

$$\frac{xRy, y : A, x : \Box A, \Gamma \vdash z : C}{xRy, x : \Box A, \Gamma \vdash z : C}\ \Box L \qquad \frac{xRy, \Gamma \vdash y : A}{\Gamma \vdash x : \Box A}\ \Box R \qquad \frac{xRy, y : A, \Gamma \vdash z : C}{x : \Diamond_i A, \Gamma \vdash z : C}\ \Diamond_i L$$

$$\frac{xRy, \Gamma \vdash y : A}{xRy, \Gamma \vdash x : \Diamond_i A}\ \Diamond_i R \qquad \frac{xRy, y : A, \Gamma \vdash x : \bot}{x : \Diamond_c A, \Gamma \vdash x : \bot}\ \Diamond_c L \qquad \frac{x : \Box\neg A, \Gamma \vdash x : \bot}{\Gamma \vdash x : \Diamond_c A}\ \Diamond_c R$$

Fig. 2. Ecumenical modal system labEK. In rules $\Box R$, $\Diamond_i L$, $\Diamond_c L$, the eigenvariable y is fresh.

$$\begin{array}{lcl} [\Box A]^e_x & := & \forall y(R(x,y) \rightarrow_i [A]^e_y) \\ [\Diamond_i A]^e_x & := & \exists_i y(R(x,y) \wedge [A]^e_y) \\ [\Diamond_c A]^e_x & := & \exists_c y(R(x,y) \wedge [A]^e_y) \end{array}$$

Observe that, due to equivalence (iii), we have $\Diamond_c A \leftrightarrow_i \neg\Box\neg A$. We will denote by EK the Ecumenical modal logic meta-logically characterized by LEci via $[\cdot]^e_x$.

4 A labeled system for EK

One of the advantages of having an Ecumenical framework is that some well known classical/intuitionistic systems arise as fragments [2]. In the following, we will seek such systems by proposing a labeled sequent system for Ecumenical modalities.

The basic idea behind labeled proof systems for modal logic is to internalize elements of the associated Kripke semantics (namely, the worlds of a Kripke structure and the accessibility relation between them) into the syntax. *Labeled sequents* have the form $\Gamma \vdash z : C$, where Γ is a multiset containing *labeled formulas* of the form $x : A$ and *relational atoms* of the form xRy, where x, y range over a set of variables and A is a modal formula.

Following [4], the meta-logical soundness and completeness theorems are proved via a translation between rule applications in labEK and derivations in LEci.

Theorem 4.1 *Let Γ be a multiset of labeled modal formulas and denote $[\Gamma] = \{R(x,y) \mid xRy \in \Gamma\} \cup \{[B]^e_x \mid x : B \in \Gamma\}$. The following are equivalent:*

1. $\Gamma \vdash x : A$ *is provable in* labEK.
2. $[\Gamma] \Rightarrow [A]^e_x$ *is provable in* LEci.

Finally, observe that, when restricted to the intuitionistic and neutral operators, labEK matches *exactly* Simpson's sequent system $\mathcal{L}_{\Box\Diamond}$ [4].

5 Discussion and conclusion

This is a short version of the text available at `https://arxiv.org/abs/2005.14325`. There, the interested reader may find: all the proofs; an axiomatic and semantical interpretation of Ecumenical modalities; an extension of the discussion to relational systems with the usual restrictions on the relation in the Kripke model; and a discussion about logical Ecumenism in general.

We end the present text by noting that there is an obvious connection between the Ecumenical approach and Gödel-Gentzen's double-negation translations of classical logic into intuitionistic logic. This could lead to the erroneous conclusion that the ecumenical refinement of classical logic is *essentially* the same refinement produced by such translation. But, on a closer inspection, the ecumenical approach *is not* essentially Gödel-Gentzen translation:

(i) Classical mathematical practice does not require that every occurrence of $\vee$ in real mathematical proofs be replaced by its Gödel-Gentzen translation: there is no reason to translate the occurrence of $\vee$ in the theorem $(A \to (A \vee B))$. Given that the Gödel-Gentzen translation function systematically and globally eliminates every occurrence of $\vee$ and $\exists$ from the language of classical logic, one may say that the ecumenical system reflects more faithfully the "local" necessary uses of classical reasoning.

(ii) The Gödel-Gentzen constructive refinement is based on a (systematic and total) translation function between the language of classical logic and the language of intuitionistic logic, while the ecumenical refinement considers how classical theorems are proved.

That is, the ecumenical refinement "interpolates" the Gödel-Gentzen-translation function. And this is extended, in our work, to reasoning with modalities.

References

[1] Blackburn, P., M. d. Rijke and Y. Venema, "Modal Logic," Cambridge Tracts in Theoretical Computer Science, Cambridge University Press, 2001.

[2] Pimentel, E., L. C. Pereira and V. de Paiva, *An ecumenical notion of entailment* (2020), accepted to Synthese.
URL `https://doi.org/10.1007/s11229-019-02226-5`

[3] Prawitz, D., *Classical versus intuitionistic logic*, Why is this a Proof?, Festschrift for Luiz Carlos Pereira **27** (2015), pp. 15–32.

[4] Simpson, A. K., "The Proof Theory and Semantics of Intuitionistic Modal Logic," Ph.D. thesis, College of Science and Engineering, School of Informatics, University of Edinburgh (1994).

Conditional Logic is Complete for Convexity in the Plane (Summary)

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Abstract

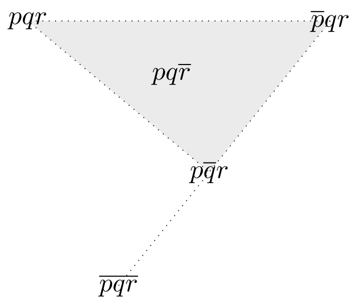
We prove completeness of preferential conditional logic with respect to convexity over finite sets of points in the Euclidean plane. A conditional is defined to be true in a finite set of points if all extreme points of the set interpreting the antecedent satisfy the consequent. Equivalently, a conditional is true if the antecedent is contained in the convex hull of the points that satisfy both the antecedent and consequent. Our result is then that every consistent formula without nested conditionals is satisfiable in a model based on a finite set of points in the plane. The proof relies on a result by Richter and Rogers showing that every
finite abstract convex geometry can be represented by convex polygons in the plane.

Keywords: conditional logic, convex geometry, nonmonotonic consequence relations.

1 Introduction

Preferential conditional logic was introduced by Burgess [3] and Veltman [17] to axiomatize the validities of the conditional with respect to a semantics in models based on preorder. In this semantics a conditional $\varphi \rightsquigarrow \psi$ is true with respect to a preorder over a finite set of worlds if the consequent ψ is true at all worlds that are minimal in the order among the worlds at which the antecedent φ is true. Both Burgess and Veltman observe that completeness already holds for partial orders instead of just preorders.

Preferential conditional logic has also been shown to be complete with respect to semantic interpretations that are quite different from the semantics in terms of partial orders. Most notable are the interpretation of validity of inferences between conditionals as preservation of high conditional probability [1,5] and premise semantics, where the conditional is interpreted relative to a premise set. A premise set is a family of sets of worlds, thought of as propositions that encode relevant background information from the linguistic context [16,9]. In the paper summarized here [10] we provide yet another interpretation to preferential conditional logic. We show that it is complete with respect to convexity over finite sets of points in the Euclidean plane. This places conditional logic into the tradition of modal logics with a natural spatial semantics, most famous of which is the completeness of S4 with respect to the topology of the real line [13].



true conditionals:
$(p \vee q) \rightsquigarrow r$
$(\neg p \vee \neg q) \rightsquigarrow \neg p$
$\top \rightsquigarrow (q \leftrightarrow r)$

false conditionals:
$p \rightsquigarrow r$
$\neg r \rightsquigarrow \neg q$
$\top \rightsquigarrow r$

Fig. 1. A finite set of points in the plane and examples of conditionals that are true or false relative to this set of points.

2 Evaluating the conditional in the plane

To illustrate our semantics consider the finite set of points in Figure 1. Think of these points as satisfying propositional letters as indicated in their label. For instance the point $\overline{p}qr$ in the upper right corner satisfies q and r but not p. Our semantics is such that a conditional $\varphi \rightsquigarrow \psi$ is true relative to such a set of points if the set of points at which φ is true is completely contained in the convex hull of the set of points at which both φ and ψ are true. Recall that a convex set is a set that for any two points in the set also contains the complete line segment between these points. Intuitively, these are the sets without holes or dents. The convex hull of a set is the least convex set that contains the set. In Figure 1 the conditional $(p \vee q) \rightsquigarrow r$ is true because all points at which $p \vee q$ is true are contained in the convex hull of the the points where $p \vee q$ and r are both true, which is the shaded area in the figure. The conditional $p \rightsquigarrow r$ is however not true in the example because the point $pq\overline{r}$ satisfies p but it not contained in the convex hull of the points pqr and $p\overline{q}r$, which are all the points that satisfy p and q.

An equivalent formulation of our semantic clause is that a conditional $\varphi \rightsquigarrow \psi$ is true if the consequent ψ is true at all the extreme points of the set of points where the antecedent φ is true. An extreme point of some set is a point in the set that is not in the convex hull of all the other points from the set. Intuitively, the extreme points of some set are the outermost points of that set. In the example from Figure 1 we have that pqr, $\overline{p}qr$ and $p\overline{q}r$ are the extreme points of the shaded area. On the other hand $pq\overline{r}$ is not an extreme point of the shaded area because it is in the convex hull of the points pqr, $\overline{p}qr$ and $p\overline{q}r$. Note that in this formulation of the semantic clause for a conditional $\varphi \rightsquigarrow \psi$ the extreme points of the set of points satisfying the antecedent φ play a role that is analogous to the minimal φ-worlds in the order semantics.

Our semantics is only defined for formulas that do not contain nested conditionals and in which all propositional letters occur in the scope of a conditional. It is possible to overcome this restriction but this would not significantly influence the axiomatic questions that this paper is concerned with.

The main completeness result of our paper can be formulated as follows: All finite constellation of points in the plane of the kind shown in Figure 1 satisfy all the theorems in preferential conditional logic and every formula that is not a theorem of the logic is false in some such constellation.

3 The proof of completeness

The completeness proof for the semantics in the plane consists of two steps:

(i) We first observe that preferential conditional logic is complete for a semantic in models based on finite abstract convex geometries.

(ii) We then show that every finite abstract convex geometry can be represented by a finite set of points in the plane in such a way that all true formulas of conditional logic are preserved.

From these two steps we obtain our completeness result because by the first step every consistent formula φ is true in some finite model based on abstract convex geometries and by the second step this model can be transformed into a concrete model of φ that is based on a finite set of points in the plane. We now describe these two steps in greater detail.

3.1 Abstract convex geometries

In the first step we make use of the notion of a convex geometry [4,8,2]. Formally, convex geometries are families of sets that are closed under arbitrary intersections and have the anti-exchange property, which is a separation property that is reminiscent of the T_0 separation property in topology. Convex geometries are a combinatorial abstraction of the notion of a convex set in Euclidean spaces, such as the Euclidean plane. This is somewhat analogous to how topological spaces are an abstraction from the notions of open and closed sets in Euclidean spaces. The convex sets in any subspace of an Euclidean space form a convex geometry. But it is not the case that every abstract convex geometry, or even every finite abstract convex geometry, is isomorphic to a subspace of some Euclidean space. An easy way to see this is that in any Euclidean space all singleton sets are convex, which is not enforced by the definition of a convex geometry.

One can view the semantics in convex geometries as a generalization of the order semantics over partial orders. The family of upwards closed sets in any partial order form a convex geometry. Moreover, a conditional is true relative to a given partial order if and only if it is also true in the convex geometry of all upwards closed sets in the order.

The semantics in abstract convex geometries can also be seen as a further development of premise semantics. The convex sets in our semantics play the role of the complements of the sets of worlds in the premise set of premise semantics. There is, however, a crucial difference in the semantic clause with which a conditional is interpreted in a family of sets of worlds. Motivated by linguistic considerations premise semantics uses a quite sophisticated semantic clause that is insensitive to closing the family of sets under intersections. In

[14,6] it is observed that for developing proof systems for preferential conditional logic it is beneficial to lift the implicit assumption that family of sets of worlds, relative to which the conditional is evaluated, is closed under intersections. To achieve this they use a simplified semantic clause from [11] that is sensitive to closure under intersections. When one uses the conditional with this semantic clause relative to a family of sets of worlds that is not closed under intersection different formulas turn out to be true than would be true relative to the same family of sets of worlds using the semantic clause from premise semantics. Thus, it is helpful to distinguish this new setting from premise semantics and call it neighborhood semantics.

This neighborhood semantics is also the starting point for the categorical correspondence in [12]. This paper establishes a correspondence between finite Boolean algebras with additional structure that encodes non-nested preferential conditional logic and families of subsets of the atoms of these algebras. To obtain a well-behaved correspondence it is necessary to allow for families of sets that are not closed under intersections. However, one can require closure under unions and a separation property that is dual to the anti-exchange property mentioned above. If one then considers the complements of all the sets in a such a family of sets then one obtains a family that is a convex geometry.

3.2 Representation of convex geometries in the plane

The second step of proof is to show that for every abstract convexity there is a finite subspace of the plane that satisfies the same formulas in conditional logic. This step is not trivial because, as we already explained above, not every finite convex geometry is isomorphic to a subspace of some Euclidean space. However, following [7], there has recently been a lot of literature on representing finite convex geometries inside of Euclidean spaces using more complex constructions than just selecting a subspace. For the proof of completeness we make use of one such representation result by [15]. Their construction shows that every finite convex geometry is isomorphic to the convexity over a set of polygons in the plane, such that every point in the original convex geometry corresponds to a whole polygon in this set. This representation is such that the extreme points of any two polygons in the set of polygons are disjoint. One can thus define a function that maps an extreme point of some polygon in the set to the point in the original convex geometry that the polygon is representing. The domain of this function can be considered to be the finite subspace of the plane consisting of all the points that are an extreme point of one of the polygons. The crucial insight is then that this function is a strong morphism of convex geometries in a sense defined in [12], which guarantees the preservation of true formulas in conditional logic.

4 Limitative results and open questions

The completeness result for finite sets of points in the plane cannot be improved to a completeness result with respect to finite set of points on the real line. The reason is that the line validates additional formulas that are not theorems of preferential conditional logic. As an example consider the formula

$$\delta_2 = (p \vee q \vee r \rightsquigarrow s) \to (p \vee q \rightsquigarrow s) \vee (p \vee r \rightsquigarrow s) \vee (q \vee r \rightsquigarrow s).$$

To get an intuition for why δ_2 is valid in every finite set of points on the line consider the extreme points of the set of all points satisfying $p \vee q \vee r$. There are at most two such extreme points, namely the maximal and minimal elements of this set in the standard order on the reals. Now these two worlds are also the extreme points of at least one of the sets interpreting $p \vee q$, $p \vee r$ and $q \vee r$. This example rises the question what axioms are necessary to obtain completeness for the conditional logic of the real line.

A further open question is whether it is possible to prove completeness of preferential conditional logic with respect to infinite sets of points in the plane. The semantic clause taken from neighborhood semantics can also be used on infinite convex geometries, but most of the methods used in our completeness proof apply only to the finite case.

References

[1] Adams, E., “The Logic of Conditionals: An Application of Probability to Deductive Logic,” Springer, 1975.

[2] Adaricheva, K. and J. B. Nation, *Convex geometries*, in: G. Grätzer and F. Wehrung, editors, *Lattice Theory: Special Topics and Applications*, Springer, 2016 pp. 153–179.

[3] Burgess, J., *Quick completeness proofs for some logics of conditionals*, Notre Dame Journal of Formal Logic **22** (1981), pp. 76–84.

[4] Edelman, P. H. and R. E. Jamison, *The theory of convex geometries*, Geometriae Dedicata **19** (1985), pp. 247–270.

[5] Geffner, H., *High-probabilities, model-preference and default arguments*, Minds and Machines **2** (1992), pp. 51–70.

[6] Girlando, M., S. Negri and N. Olivetti, *Uniform labelled calculi for preferential conditional logics based on neighbourhood semantics* (2020), on arXiv.

[7] Kashiwabara, K., M. Nakamura and Y. Okamoto, *The affine representation theorem for abstract convex geometries*, Computational Geometry **30** (2005), pp. 129–144.

[8] Korte, B., L. Lovász and R. Schrader, “Greedoids,” Springer, 1991.

[9] Kratzer, A., *Partition and revision: The semantics of counterfactuals*, Journal of Philosophical Logic **10** (1981), pp. 201–216.

[10] Marti, J., *Conditional logic is complete for convexity in the plane* (2020), on arXiv.

[11] Marti, J. and R. Pinosio, *Topological semantics for conditionals*, in: V. Punčochář and P. Švarný, editors, *The Logica Yearbook 2013*, College Publications, 2014 pp. 115–128.

[12] Marti, J. and R. Pinosio, *A discrete duality between nonmonotonic consequence relations and convex geometries*, Order (2019).

[13] McKinsey, J. C. C. and A. Tarski, *The algebra of topology*, Annals of Mathematics (1944), pp. 141–191.

[14] Negri, S. and N. Olivetti, *A sequent calculus for preferential conditional logic based on neighbourhood semantics*, in: H. D. Nivelle, editor, *Automated Reasoning with Analytic Tableaux and Related Methods, TABLEAUX 2015*, Springer, 2015 pp. 115–134.

[15] Richter, M. and L. G. Rogers, *Embedding convex geometries and a bound on convex dimension*, Discrete Mathematics **340** (2017), pp. 1059–1063.

[16] Veltman, F., *Prejudices, presuppositions, and the theory of counterfactuals*, in: J. Groenendijk and M. Stokhof, editors, *Proceedings of the Amsterdam Colloquium on Montague grammar and related topics*, Amsterdam Papers in Formal Grammar **1**, 1976, pp. 248–282.

[17] Veltman, F., “Logics for Conditionals,” Ph.D. thesis, University of Amsterdam (1985).

A W-flavoured series of interpretability principles

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Abstract

While attempting to prove that the logic **IL**WR is modally complete, we found a new series of interpretability principles. In this short paper we sketch the proofs that the series is arithmetically sound, show that principles are valid in ordinary **IL**WR-frames, and evaluate the possible impact of our results.

Keywords: Formalised interpretability, interpretability logic, modal logic.

1 Introduction

Interpretability logics are propositional modal logics extending provability logics with a binary modality $\rhd$ denoting formal interpretability over some base theory T. We shall mostly be interested in so-called *sequential theories*. These theories can code pairs of objects and as such the natural numbers can naturally be embedded in them together with coding machinery for syntax so that indeed the notion of interpretability can be formalised. Then, for some sequential base theory T, the expression $A \rhd B$ will stand for "T together with some arithmetical reading of B is interpretable in T together with the arithmetical reading of A".

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The language of interpretability logics extends the basic (mono)modal language with formulas of form $A \rhd B$:

$$A \ ::= \ \bot \mid \mathsf{Var} \mid \mathsf{A} \to \mathsf{A} \mid \Box \mathsf{A} \mid \mathsf{A} \rhd \mathsf{A}$$

where Var generates a countable set of propositional variables. Since usually we take the logic **IL** as our base logic, and $\mathbf{IL} \vdash \Box\neg A \leftrightarrow A \rhd \bot$, we can choose to leave the symbol $\Box$ out of the language.

We adopt a reading convention due to Dick de Jongh that allows us to write fewer brackets. The precedence is such that the strongest binding symbols are $\neg$, $\Box$ and $\Diamond$ which all bind equally strong. Next come $\wedge$ and $\vee$, followed by $\rhd$ and the weakest connectives are $\to$ and $\leftrightarrow$. Thus, for example, $A \rhd B \to A \wedge \Box C \rhd B \wedge \Box C$ will be short for $(A \rhd B) \to \big((A \wedge \Box C) \rhd (B \wedge \Box C)\big)$.

Given a sequential theory T, the logic $\mathbf{IL}(T)$ is the set of modal formulas whose so-called arithmetical interpretations are provable in T. An arithmetical formula is an arithmetical interpretation of A if it is obtained from A by substituting propositional variables with sentences, and the operator $\rhd$ with the interpretability predicate. There are multiple plausible choices for the notion of an *interpretability predicate.* Unless stated otherwise, we are talking about *theorems interpretability*: $\mathrm{Int}(A, B)$ stands for "there is a translation function $*$ such that for all C, if $T + B \vdash C$ then $T + A \vdash C^*$". Here $*$ is any translation, a function that preserves structure up to quantifier relativisation (see e.g. [8] for details).

Next we turn to relational semantics. The results in this paper rely on the (ordinary or regular) Veltman semantics. The future work, and indeed the motivation for this paper, is centred around the notion of *generalised Veltman semantics.* So let us define both the regular and generalised Veltman semantics.

Definition 1.1 A *generalised Veltman frame* $\mathfrak{F}$ is a structure $\langle W, R, \{S_w : w \in W\}\rangle$, where W is a non-empty set, R is a transitive and converse well-founded binary relation on W and for all $w \in W$ we have:

a) $S_w \subseteq R[w] \times (\mathcal{P}(R[w]) \setminus \{\emptyset\})$;
b) S_w is quasi-reflexive: wRu implies $uS_w\{u\}$;
c) S_w is quasi-transitive: if uS_wV and vS_wZ_v for all $v \in V$, then $uS_w(\bigcup_{v\in V} Z_v)$;
d) if $wRuRv$, then $uS_w\{v\}$;
e) monotonicity: if uS_wV and $V \subseteq Z \subseteq R[w]$, then uS_wZ.

A *generalised Veltman model* is a quadruple $\mathfrak{M} = \langle W, R, \{S_w : w \in W\}, V\rangle$, where the first three components form a generalised Veltman frame and where V is a valuation mapping propositional variables to subsets of W. The forcing relation $\mathfrak{M}, w \Vdash A$ is defined in the expected way together with the following:

$$w \Vdash A \rhd B \ :\Longleftrightarrow \ \forall u \ \Big(wRu \ \& \ u \Vdash A \Rightarrow \exists V(uS_wV \ \& \ V \Vdash B)\Big).$$

We write $V \Vdash B$ as short for $(\forall\, v{\in}V)\ v \Vdash B$.

To save some space, we define regular Veltman semantics by stipulating that a generalised model is an ordinary model if whenever uS_wV, the set V is a singleton (i.e. $V = \{v\}$ for some v), and we exclude monotonicity.

By an **IL**X-frame we mean (a regular, if not stated otherwise) frame such that no theorem of **IL**X can be refuted using this frame. We say that a logic is complete w.r.t. (generalised) Veltman semantics if for any non-theorem A there is a (generalised) Veltman model satisfying A in one of its worlds.

In [7] the logic known as **IL**R ($\mathbf{IL} + A \rhd B \to \neg(A \rhd \neg C) \rhd B \wedge \Box C$) was proven to be modally complete (w.r.t. generalised semantics); and another, known as **IL**W ($\mathbf{IL} + A \rhd B \to A \rhd B \wedge \Box\neg A$), was known to be modally complete much earlier [1]. Problems occurred while trying to prove that the combination of these two logics, **IL**WR, is modally complete (see [2] for the statement of the problem and a discussion on how to overcome the problem). At the moment we believe **IL**WR is modally complete if it can prove principles contained in a certain ("W-flavoured") series of principles. In this short paper we will only define this series and prove that the principles contained are arithmetically valid. We do not yet know e.g. if they are independent from other known principles.

A major open problem in the field is to characterise **IL**(All), the intersection of interpretability logics of all sequential theories T. The search for **IL**(All) benefited from exploring modal semantics, in this case the so-called Veltman semantics (e.g. [3]). This is our motivation too. For definitions and other details concerning formalised interpretability please see the literature, e.g. [8].

The semi-formal modal logic CuL was introduced in [6]. The system is based on a richer modal language than the language of interpretability logics: modal operators are allowed to have a variable in their superscript. The intended arithmetical interpretation of this variable is a definable cut. In case the cut in question is the identity cut, we will just omit it. Various principles in **IL**(All) allow for an arithmetical soundness proof using CuL. (See [6], [4], and the forthcoming [5].) Due to size constraints, we will display some essential ingredients of the system without further comments referring the diligent reader to [6]:

$(\to)^{\mathsf{J}}$	$\vdash \Box^I A \to \Box A$
$\mathsf{L}_1^{\mathsf{J}}$	$\vdash \Box^I(A \to B) \to (\Box^I A \to \Box^I B)$
$\mathsf{L}_2^{\mathsf{J}}$	$\vdash \Box^I A \to \Box^I \Box^J A$
$\mathsf{L}_3^{\mathsf{J}}$	$\vdash \Box^I(\Box^J A \to A) \to \Box^I A$
$\mathsf{J}_1^{\mathsf{J}}$	$\vdash \Box(A \to B) \to A \rhd B$
$\mathsf{J}_5^{\mathsf{J}}$	$\vdash \Diamond^J A \rhd A$
$\mathsf{Nec}^{\mathsf{J}}$	$\vdash A \Rightarrow \vdash \Box^I A$
M^{J}	$\Gamma, (A \wedge \Box^J C \rhd B \wedge \Box^{J'} C) \vdash D \Rightarrow \Gamma, A \rhd B \vdash D$
	Here J is a variable not occuring in Γ, A, B, D and $J \neq J'$

Of course we also use regular principles like $\mathsf{J}_2 : (A \rhd B) \wedge (B \rhd C) \to A \rhd C$, $\mathsf{J}_3 : (A \rhd C) \wedge (B \rhd C) \to A \vee B \rhd C$ and $\mathsf{J}_4 : A \rhd B \to (\Diamond A \to \Diamond B)$.

2 A W-flavoured series of principles

We define the series of principles $(\mathsf{W}_n)_{n\in\omega}$ by stating $\mathsf{W_0} := \mathsf{W} = A \rhd B \to A \rhd B \wedge \Box\neg A$ and for $n > 0$:

$$\begin{aligned} \mathsf{U}_n &:= \Diamond C_{n-1} \vee \cdots \vee \Diamond C_1; \\ \mathsf{V}_1 &:= A; \\ \text{for } n>1: \quad \mathsf{V}_n &:= \neg(C_{n-1} \rhd \Diamond A \vee B_{n-1} \vee \mathsf{U}_{n-1} \to \mathsf{V}_{n-1} \rhd B_{n-1}); \\ \text{for } n>0: \quad \mathsf{W}_n &:= A \rhd \Diamond A \vee B_n \vee \mathsf{U}_n \to \mathsf{V}_n \rhd B_n. \end{aligned}$$

Thus, the first few principles are ($\mathsf{W_0}$ actually being equivalent to $\mathsf{W_1}$):

$$\begin{aligned} &\mathsf{W}_1: A \rhd \Diamond A \vee B_1 \to A \rhd B_1; \\ &\mathsf{W}_2: A \rhd \Diamond A \vee B_2 \vee \Diamond C_1 \to \neg(C_1 \rhd \Diamond A \vee B_1 \to A \rhd B_1) \rhd B_2; \\ &\mathsf{W}_3: A \rhd \Diamond A \vee B_3 \vee \Diamond C_2 \vee \Diamond C_1 \to \\ &\quad \to \neg(C_2 \rhd \Diamond A \vee B_2 \vee \Diamond C_1 \to \neg(C_1 \rhd \Diamond A \vee B_1 \to A \rhd B_1) \rhd B_2) \rhd B_3. \end{aligned}$$

We omit the proofs of the following two lemmas.

Lemma 2.1 *Let $n \in \omega \setminus \{0\}$. Suppose $\Box(A \to \bigvee_{1\le i\le n-1} \Diamond^K \neg C_i)$ and $C_{n-1} \rhd \Diamond A \vee B_{n-1} \vee \mathsf{U}_{n-1}$. Then for some cut J the following holds:*

$$C_{n-1} \wedge \bigwedge_{1\le i\le n-2} \Box^J \neg C_i \rhd B_{n-1}$$

Lemma 2.2 *For all cut variables K and all $n \in \omega \setminus \{0\}$,*

$$\vdash \mathsf{V}_n \rhd A \wedge \bigwedge_{1\le i\le n-1} \Box^K \neg C_i.$$

Proposition 2.3 *For all $n \in \omega \setminus \{0\}$, $\vdash \mathsf{W}_n$, i.e.*

$$\vdash A \rhd \Diamond A \vee B_n \vee \mathsf{U}_n \to \mathsf{V}_n \rhd B_n$$

Proof Suppose $A \rhd \Diamond A \vee B_n \vee \mathsf{U}_n$. Applying the principle W, $A \rhd B_n \vee \mathsf{U}_n$. Then there is a cut K such that

$$A \wedge \bigwedge_{1\le i\le n-1} \Box^K \neg C_i \rhd (B_n \vee \mathsf{U}_n) \wedge \bigwedge_{1\le i\le n-1} \Box\neg C_i.$$

By unpacking U_n we see that

$$A \wedge \bigwedge_{1\le i\le n-1} \Box^K \neg C_i \rhd B_n \wedge \bigwedge_{1\le i\le n-1} \Box\neg C_i.$$

In particular,

$$A \wedge \bigwedge_{1\le i\le n-1} \Box^K \neg C_i \rhd B_n.$$

Lemma 2.2 implies

$$\mathsf{V}_n \rhd A \wedge \bigwedge_{1\leq i\leq n-1} \Box^K \neg C_i.$$

Applying J2 gives $\mathsf{V}_n \rhd B_n$, as required. □

Theorem 2.4 *For $n \in \omega$, the principle W_n is valid in* **IL**WR*-frames.*

Proof Omitted. □

3 Conclusion, status and future work

Let us briefly comment on the status of the new series. At the moment we don't have answers to the following three questions: (1) Is $\{\mathsf{W}_n\}_{n\in\omega}$ valid on generalised **IL**WR-frames?; (2) do we have $\mathbf{IL}\mathsf{WR} \Vdash W_n$ for all $n \in \omega$?; (3) do we have $\mathbf{IL}\mathsf{W}\{\mathsf{R_k R^m}\}_{\mathsf{k,m}\in\omega} \Vdash W_n$ for all $n \in \omega$?

If (1), we have (unpublished) modal completeness of $\mathbf{IL}\{\mathsf{W_n}\}_{\mathsf{n}\in\omega}$ w.r.t. generalised semantics. This is a strictly stronger system than **IL**W and **IL**R, and so would be the strongest system yet for which we have modal completeness. If (1) and (2), then $\mathbf{IL}\mathsf{WR} = \mathbf{IL}\mathsf{W_n}$, and so we also have completeness of **IL**WR w.r.t. generalised semantics.
If (1) and not (2) are the case, in addition we have incompleteness of **IL**WR.

If (1) is not the case, then $\{\mathsf{W}_n\}_{n\in\omega}$ is independent of **IL**WR. If (1) is not the case and (3) is the case, with additional work we might still be able to prove completeness of $\mathbf{IL}\{\mathsf{W_n}\}_{\mathsf{n}\in\omega}$ w.r.t. generalised semantics. If neither (1) or (3), we have a (strictly) better lower bound of **IL**(All): the logic $\mathbf{IL}\{\mathsf{W_n R_k R^m}\}_{\mathsf{k,m,n}\in\omega}$.

References

[1] de Jongh, D. and F. Veltman, *Modal completeness of IL*W, in: J. Gerbrandy, M. Marx, M. Rijke and Y. Venema, editors, *Essays dedicated to Johan van Benthem on the occasion of his 50th birthday*, Amsterdam University Press, Amsterdam, 1999 .

[2] Goris, E., M. Bílková, J. Joosten and L. Mikec, *Assuring and critical labels for relations between maximal consistent sets for interpretability logics* (2020).
URL `https://arxiv.org/2003.04623`

[3] Goris, E. and J. Joosten, *A new principle in the interpretability logic of all reasonable arithmetical theories*, Logic Journal of the IGPL **19** (2011), pp. 14–17.

[4] Goris, E. and J. Joosten, *Two new series of principles in the interpretability logic of all reasonable arithmetical theories*, The Journal of Symbolic Logic **85** (2020), pp. 1–25.

[5] Joosten, J., L. Mikec and A. Visser, *Feferman axiomatisations, definable cuts and principles of interpretability*, forthcoming (2020).

[6] Joosten, J. and A. Visser, *How to derive principles of interpretability logic, A toolkit*, in: J. v. Benthem, F. Troelstra, A. Veltman and A. Visser, editors, *Liber Amicorum for Dick de Jongh*, Intitute for Logic, Language and Computation, 2004 Electronically published, ISBN: 90 5776 1289.

[7] Mikec, L. and M. Vuković, *Interpretability logics and generalised Veltman semantics*, The Journal of Symbolic Logic (to appear).

[8] Visser, A., *An overview of interpretability logic*, in: M. Kracht, M. d. Rijke and H. Wansing, editors, *Advances in modal logic '96*, CSLI Publications, Stanford, CA, 1997 pp. 307–359.

Independent worm battles

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Abstract

In this extended abstract we study Beklemishev's combinatorial principle *Every Worm Dies*, EWD from [2]. This principle arises from considering a sequence of modal formulas, the finiteness of which is not provable in Peano Arithmetic, being equivalent to the one-consistency of PA. We show that this theorem can be generalised in a straight-forward fashion to natural fragments of PA. Furthermore, we comment on our progress to extending the framework to fragments of second order arithmetic, most notably ACA.

Keywords: Provability logics, independence results, ordinal analysis.

1 Preliminaries

The polymodal provability logic **GLP** has turned out a versatile logic since special elements –the so-called *worms*– in there can be interpreted in many ways: elements of a logic, words over an infinite alphabet, special fragments of arithmetic, Turing progressions, worlds in a special model for the closed fragment of **GLP**, and also ordinals. Due to these many interpretations of worms, Beklemishev could give ([1]) an ordinal analysis of PA and related systems. As a consequence, he could formulate a combinatorial principle about worms that is true yet independent of PA.

In the recent paper [3], Beklemishev and Pakhomov extend the method of ordinal analysis via provability logics to predicative systems of second order arithmetic. It is important to investigate if said analysis also comes with the expected regular side-products as classification of provably total recursive functions, consistency proofs, and independent combinatorial principles. This paper can be seen as some first explorations in this direction.

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Definition 1.1 For Λ an ordinal, the logic $\mathbf{GLP}_\Lambda$ is the propositional modal logic with a modality $[\alpha]$ for every $\alpha < \Lambda$. Each $[\alpha]$ modality satisfies the **GL** identities given by all tautologies, distribution axioms $[\alpha](\varphi \to \psi) \to ([\alpha]\varphi \to [\alpha]\psi)$, Löb's axiom scheme $[\alpha]([\alpha]\varphi \to \varphi) \to [\alpha]\varphi$ and the rules modus ponens and necesitation $\varphi/[\alpha]\varphi$. The interaction between modalities is governed by two schemes, monotonicity $[\beta]\varphi \to [\alpha]\varphi$ and, negative introspection $\langle\beta\rangle\varphi \to [\alpha]\langle\beta\rangle\varphi$ where in both schemes it is required that $\beta < \alpha < \Lambda$.

The closed fragment of $\mathbf{GLP}_\Lambda$ suffices for ordinal analyses and worms are the backbone of it.

Definition 1.2 The class of *worms* of $\mathbf{GLP}_\Lambda$ is denoted $\mathbb{W}^\Lambda$ and defined by $\top \in \mathbb{W}^\Lambda$ and, $A \in \mathbb{W}^\Lambda \wedge \alpha < \Lambda \Rightarrow \langle\alpha\rangle A \in \mathbb{W}^\Lambda$. By $\mathbb{W}^\Lambda_\alpha$ we denote the set of worms where all occurring modalities are at least α. Worms $A, B \in \mathbb{W}^\Lambda$ allow orderings $<_\alpha$ for any $\alpha < \Lambda$ by defining $A <_\alpha B :\Leftrightarrow \mathbf{GLP}_\Lambda \vdash B \to \langle\alpha\rangle A$. We define the α-head h_α of A inductively: $h_\alpha(\top) := \top$ and $h_\alpha(\langle\beta\rangle A) := \top$ if $\beta < \alpha$ and $h_\alpha(\langle\beta\rangle A) := \langle\beta\rangle h_\alpha(A)$ otherwise. Likewise, we define the α-remainder r_α of A as $r_\alpha(\top) := \top$ and, $r_\alpha(\langle\beta\rangle A) := \langle\beta\rangle A$ if $\beta < \alpha$ and $r_\alpha(\langle\beta\rangle A) := r_\alpha(A)$ otherwise. We define the head h and remainder r of $\langle\alpha\rangle A$ as $h(\langle\alpha\rangle A) := h_\alpha(\langle\alpha\rangle A)$ and $r(\langle\alpha\rangle A) := r_\alpha(\langle\alpha\rangle A)$. Further, $h(\top) := r(\top) := \top$.

The modalities can be linked to arithmetic by interpreting $\langle n\rangle\varphi$ as the finitely axiomatisable scheme $\mathrm{RFN}_{\Sigma_n}(\mathrm{EA}+\varphi^*) := \{\Box_{\mathrm{EA}+\varphi^*}\sigma \to \sigma \mid \sigma \in \Sigma_n\}$ where EA denotes Kalmar elementary arithmetic which is essentially induction for bounded arithmetical formulas together with an axiom stating that the graph of exponentiation defines a total function. The $\Box_{\mathrm{EA}}$ –we will often simply also write $\Box$– denotes the standard arithmetisation of formalised provability and φ^* denotes an interpretation of φ in arithmetic, mapping propositional variables to sentences, commuting with the connectives and, translating the $\langle n\rangle$ as above. The theory EA^+ is as EA now stating that superexponentiation is a total function. Simple worms relate to arithmetic via the following.

Theorem 1.3 (Leivant, Beklemishev, Kreisel, Levy) *Provably in* EA^+, *for* $n \geq 1$ *and* $*$ *arbitrary:* $\mathrm{I}\Sigma_n \equiv \mathrm{RFN}_{\Sigma_n}(\mathrm{EA}) \equiv (\langle n+1\rangle\top)^*$ *and* $\mathrm{PA} \equiv \{(\langle m\rangle\top)^* | m \in \omega\}$.

From [1] we know that $\langle \mathbb{W}^\omega_n/\equiv, <_n\rangle \cong \langle \varepsilon_0, \prec\rangle$ so that worms (modulo provable equivalence) can be used to denote ordinals. We can find analogs of fundamental sequences for ordinals by defining $Q^0_n(A) := \langle n\rangle A$; $Q^{k+1}_n(A) := \langle n\rangle\big(A \wedge Q^k_n(A)\big)$. By an easy induction on k one sees that $Q^k_n(A) <_m \langle n+1\rangle A$ for any $m \leq n$ and the sequence $Q^k_n(A)$ approximates $\langle n+1\rangle A$ in the sense of the so-called *reduction property* from [3] [1]: $\mathrm{EA} + \langle n+1\rangle A \equiv_{\Pi^0_{n+1}} \mathrm{EA} + \{Q^k_n(A)\}_{k\in\omega}$. This is provable in EA^+ so that we get the following corollary.

Theorem 1.4 (Reduction property) $\mathrm{EA}^+ \vdash \langle m\rangle\langle n{+}1\rangle A \leftrightarrow \forall k \langle m\rangle Q^k_n(A)$ *($m \leq n$).*

[3] Since for closed formulas φ^* does not depend on $*$ we will often drop the interpretation.

The sequence $Q_n^k(A)$ can be used to define decreasing ordinal sequences by defining $(\langle n+1\rangle A)\ll k\gg := Q_n^{k+1}(A)$. To make this stepping down also be defined on successor ordinals we define $(\langle 0\rangle A)\ll k\gg := A$. Of course we cannot get smaller than the minimal element so that we define $\top\ll k\gg := \top$.

The step-down function can be rewritten to get a more combinatorial flavour reminiscent of the Hydra battle. To this end we define the *chop-operator* c on worms by $c(\top):=\top; c(\langle 0\rangle A):=A$ and, $c(\langle n+1\rangle A):=\langle n\rangle A$. Now we define a stepping down function based on a combination of chopping a word and the worm growing back. For worms A and B we define the concatenation $A\star B$ via $\top\star B:=B$ and $(\langle\alpha\rangle A)\star B:=\langle\alpha\rangle(A\star B)$.

Definition 1.5 For any number k let $A[\![k]\!]:=c(A)$ for $A=\top$ or $A=0B$ and $A[\![k]\!]:=\big(c(h(A))\big)^{k+1}\star r(A)$ otherwise.

From now on we often omit the $\star$. It is easy to prove (see [2]) that for any A and k we have that $A\ll k\gg$ is **GLP**-provably equivalent to $A[\![k]\!]$. Given a worm A, we now define a decreasing sequence (strictly as long as we have not reached $\top$) by $A_0 := A$ and $A_{k+1}:=A_k[\![k+1]\!]$. We now define the principle EWD standing for *every worm dies* as an arithmetisation of $\forall A\exists k A_k{=}\top$. The principle, although true, is not provable in PA. Actually, it turns out to be provably equivalent to the one-consistency of PA.

2 Worm battles for $\mathrm{I}\Sigma_n$

By EWD^n we will refer to the principle restricted to worms of $\mathbb{W}^n$, that is, $\forall A{\in}\mathbb{W}^n\exists k A_k{=}\top$. Through a simple adaptation of [1] and [2] we will prove that EWD^{n+1} is equivalent to the one-consistency of $\mathrm{I}\Sigma_n$ for $n{>}0$. To reach the EA-proof of EWD^{n+1} from 1-Con($\mathrm{I}\Sigma_n$) we shall make use of the following rule:

Definition 2.1 By $\mathrm{TI}^R(\Pi_n, <_0\restriction\mathbb{W}^\Lambda)$ we denote the following inference rule expressing transfinite induction along the ordering of $<_0$ for Π_n-formulas φ:

$$\frac{\forall\, A{\in}\mathbb{W}^\Lambda\big(\forall\, B{<_0}A\ \varphi(B)\to\varphi(A)\big)}{\forall\, A{\in}\mathbb{W}^\Lambda\ \varphi(A)}.$$

Then, via a conversion of the similar theorem found in [1]:

Theorem 2.2 *For every* $n{>}0$, $\mathrm{EA}+1\text{-}\mathrm{Con}(\mathrm{I}\Sigma_n)$ *contains* $[\mathrm{EA}, \mathrm{TI}^R(\Pi_2, \mathbb{W}^n)]$, *that is –the extension of* EA *by one application of the* $\mathrm{TI}^R(\Pi_2,\mathbb{W}^n)$ *rule.*

And since EWD^{n+1} is a Π_2 sentence, we clearly obtain our desired result.

For independence, as in [2], we introduce an analogue of Hardy functions: Let $h_A(m)$ be[4] the least k such that $A[\![m\ldots m{+}k]\!]{=}\top$, where $A[\![m\ldots m{+}k]\!]:=A[\![m]\!]\ldots[\![m{+}k]\!]$. Given worms A and B we define the ordering $A{\trianglelefteq}B$ iff $A{=}B[\![0]\!]\ldots[\![0]\!]$ for a finite number of iterations.[5] This relation

[4] Confusion with the h_α and h function from Definition 1.2 is not possible due to different types of arguments.

[5] iff A is an initial segment of B apart from possibly the first element which should then be smaller.

gives us an easily proven, through the definitions above, monotonicity for the h_A functions:

Lemma 2.3 *If $h_B(y)$ is defined, $A \trianglelefteq B \in \mathbb{W}^\omega$ and $x \leq y$ then*

1. $\exists\, k\ B[\![m \ldots m+k]\!]{=}A$.
2. $\forall\, m{\leq}y\ \exists\, k\ B[\![n \ldots n+k]\!]{=}A[\![m]\!]$.
3. $h_A(x)$ *is defined and* $h_A(x){\leq}h_B(y)$.

Lemma 2.4 *If $A \in \mathbb{W}_1$ and $h_B(y)$ is defined, then $h_{1B}(n){>}h_B^{(n)}(n)$.*

The above are formalizable in EA. Let $f{\downarrow}$ denote $\forall x \exists y\, f(x){=}y$.

Lemma 2.5 $\mathrm{EA} \vdash \forall\, A{\in}\mathbb{W}_1^\omega\ (h_{A1111}\downarrow \to \langle 1\rangle\, A)$.

Proof. Reasoning in EA. By Löb's theorem, we can assume that

$$\forall\, A{\in}\mathbb{W}_1^\omega\ [1](h_{A1111}\downarrow \to \langle 1\rangle\, A). \tag{1}$$

If $A1111{=}1B$ then $h_{1B}{\downarrow} \;\to\; \lambda x.h_B^{(x)}(x){\downarrow}$. The function h_B is increasing, has an elementary graph and grows at least exponentially as by Lemma 2.4, $h_{111}(x){>}2^x$. So for $A{=}\top$ we have that $h_{1111}{\downarrow}$ implies the totality of 2^x_n and hence EA^+ which is known ([2]) to imply $\langle 1\rangle\top$. For A nonempty, we reason:

$$\begin{aligned}
\lambda x.h_B^{(x)}(x)\downarrow &\to \langle 1\rangle\, h_B\downarrow, && \text{a theorem of EA([2])}\\
&\to \langle 1\rangle\, \langle 1\rangle\, B, && \text{by Assumption (1)}\\
&\to \langle 1\rangle\, A.
\end{aligned}$$

If $A1111{=}B$ starts with $m{>}1$, then as before $h_B\downarrow$ implies EA^+ and,

$$\begin{aligned}
h_B\downarrow &\to \lambda x.h_{B[\![x]\!]}(x+1)\downarrow\\
&\to \forall\, n\ h_{B[\![n]\!]}\downarrow \qquad \text{by Lemma 2.3}\\
&\to \forall\, n\ h_{B[\![n+1]\!]}\downarrow\\
&\to \forall\, n\ h_{1(B[\![n]\!])}\downarrow \qquad (\text{as } 1(B[\![n]\!]){\trianglelefteq}B[\![n+1]\!])\\
&\to \forall\, n\ \lambda x.h^{(x)}_{B[\![n]\!]}(x)\downarrow\\
&\to \forall\, n\ \ \langle 1\rangle\, h_{B[\![n]\!]}\downarrow \qquad (\text{as before})\\
&\to \forall\, n\, \langle 1\rangle\, \langle 1\rangle\, (A[\![n]\!]) \qquad \text{by Assumption (1)}\\
&\to \langle 1\rangle\, A \quad (\text{by the reduction property}).
\end{aligned}$$

□

Now to prove the independence of the worm principle for $\mathrm{I}\Sigma_n$ for $n{>}0$, assume inside EA that the principle holds. We have:

$$\begin{aligned}
\mathrm{EA} \vdash \forall\, A{\in}\mathbb{W}^{n+1}\ \exists\, m\ A_m{=}\top &\to \forall\, A{\in}\mathbb{W}_1^{n+1}\ h_A\downarrow\\
&\to \forall\, k\ \ \langle 1\rangle\, \big(\langle n+1\rangle\top[\![k]\!]\big)\\
&\to \langle 1\rangle\, \langle n+1\rangle\top \qquad (\text{by the reduction property})\\
&\to 1\text{-}\,\mathrm{Con}(\mathrm{I}\Sigma_n).
\end{aligned}$$

We can make use of the reduction property since $\langle 1\rangle\, \langle n+1\rangle\top \to \langle 1\rangle\top$, which in turn implies EA^+.

3 Towards subsystems of second-order arithmetic

Following [3] we expand the language $\mathcal{L}$ of arithmetic to $\mathcal{L}_\alpha$ with a sequence of truth predicates $\{\top_\beta : \beta < \alpha\}$ satisfying the *Uniform Tarski Biconditional axioms of truth* $\mathrm{UTB}_{<\alpha}$. That is, the schema $\forall \overline{x}\ (\varphi(\overline{x}) \leftrightarrow \top_\beta(\varphi(\dot{\overline{x}})))$ for all $\beta{<}\alpha$ and $\varphi{\in}\mathcal{L}_\beta$. We will denote $\mathrm{UTB}_\alpha := \mathrm{UTB}_{<\alpha+1}$ and $\mathrm{UTB}_{\mathcal{L}} := \mathrm{UTB}_0$. Given an elementary well ordering, $(\Lambda, <)$ we expand the arithmetical hierarchy into the so-called *hyperarithmetical hierarchy* up to $\omega(1+\Lambda)$. Let $\Pi_{\omega(1+\alpha)+n} := \Pi^{\mathcal{L}_\alpha}_{n+1}(\top_\alpha)$, and $\Pi_{<\lambda} := \bigcup_{\alpha<\lambda} \Pi_\alpha$ for limit λ. This allows us to expand reflection principles to the hyperarithmetical hierarchy. To expand the reduction property towards limit ordinals, we can use the following theorem from [3].

Theorem 3.1 *Let* $\lambda = \omega(1+\alpha)$ *and* S *provably contain* $\mathrm{EA} + \mathrm{UTB}_\alpha$*. Over* $\mathrm{EA} + \mathrm{UTB}$*, we have* $\mathrm{RFN}_{\Pi_\lambda}(S) \equiv_{\Pi_{<\lambda}} \mathrm{RFN}_{\Pi_{<\lambda}}(S)$*.*

This can lead us with some candidates to choose for the *nth* entry in the fundamental sequence for $\langle\lambda\rangle A$ worms with λ a limit ordinal. It also helps in satisfying the requirements to express some theories of second-order arithmetic as a chain of reflection principles. So for instance by [3] if we let $S := \mathrm{EA}^+ + \mathrm{UTB}_{\mathcal{L}}$ then ACA is mutually interpretable with $\mathrm{PA}(\top) := S + \mathrm{RFN}_{\Pi_{<\omega 2}}(S) \equiv_{\Pi_{<\omega 2}} S + \mathrm{RFN}_{\Pi_{\omega 2}}(S)$. With this, proving $\mathrm{EWD}^{\omega 2}$ from 1-Con(PA($\top$)) can follow the steps of the existing proof in [2].

Difficulties are met in proving its independence. Specifically in providing a sufficient monotonicity with a corresponding of Lemma 2.3. The demands for which, are dictated by Lemma 2.5 with the following implications:

$$\lambda x.h_{B[\![x]\!]}(x+1)\downarrow \to \forall n\ h_{B[\![n]\!]}\downarrow,$$
$$\forall n\ h_{B[\![n+1]\!]}\downarrow \to \forall n\ h_{1B[\![n]\!]}\downarrow .$$

A restriction to the ordering relation in accordance to the demands of Lemma 2.3 appears to clear the path for ACA and perhaps second-order theories of comparable strength. As such, this paper reports on work in progress that shall be published at some point in [4].

References

[1] Beklemishev, L. D., *Provability algebras and proof-theoretic ordinals, I*, Annals of Pure and Applied Logic **128** (2004), pp. 103–124.

[2] Beklemishev, L. D., *The Worm principle*, in: Z. Chatzidakis, P. Koepke and W. Pohlers, editors, *Logic Colloquium 2002, Lecture Notes in Logic 27*, ASL Publications, 2006 pp. 75–95.

[3] Beklemishev, L. D. and F. N. Pakhomov, *Reflection algebras and conservation results for theories of iterated truth*, arXiv:1908.10302 [math.LO] (2019).

[4] Papafillipou, K., "Independent combinatoric worm principles for first order arithmetic and beyond," Master's thesis, Master of Pure and Applied Logic, University of Barcelona (forthcoming, 2020).
URL `http://diposit.ub.edu/dspace/handle/2445/133559`

Cut-free hypersequent calculi for the logics with non-standard S5-style modalities

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Abstract

This paper is devoted to the development of cut-free hypersequent calculi for the modifications of S5 having non-standard modalities: contingency, non-contingency, essence, and accident operators. As a basis for our calculi, we take Restall's cut-free hypersequent calculus for S5. We modify its rules for the aforementioned modalities. We show that all axioms and rules of Hilbert-style axiomatizations of the logics in question are provable in our hypersequent calculi. We establish soundness, completeness and cut elimination theorems for the hypersequent calculi.

Keywords: modal logic, non-contingency logic, essence logic, accident logic, hypersequent calculus, cut elimination.

1 Introduction

Sequent and hypersequent calculi for modal logics are a fruitful and well-developed area of research. Most of standard modal logics have already had cut-free sequent or hypersequent calculi. The modal logic S5 is especially significant in this sense. Although there is no a cut-free standard sequent calculus for it, there are at least eight different cut-free hypersequent calculi and several cut-free non-standard sequent calculi for it (see [1,6] for more details). But in the case of non-standard modalities (contingency, non-contingency, essence, and accidence) the situation is worse. We know only Zolin's papers [12,11] which contain *non*-cut-free sequent calculi for some non-contingency logics (in particular, for the non-contingency version of S5). Since there are a plenty of cut-free calculi for S5, we believe that this logic is an appropriate starting point for the development of cut-free hypersequent calculi for the modal logics having non-standard modalities. Hereafter we use Restall's [9] hypersequent calculus for S5, since it is one of the simplest calculi for it.

Let us say a few words about the history of the study of non-standard modalities. Although the philosophical discussion about contingency and non-

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contingency goes back centuries, the formal presentation of contingency and non-contingency logics is due to Montgomery and Routley [7,8]. In particular, they present several axiomatizations for S5-style contingency and non-contingency logics. Essence logics were developed by Fine [3,4]. The essence modality means that "the proposition A is essentially true", i.e. "if A is true, then it is necessarily true". The formal treatment of the accident modality was done by Small [10] in the context of Gödel's ontological argument. This modality means that "although A is true, it is not necessarily true".

The structure of the paper is as follows. In Section 2, we describe the semantics and axiom systems for S5 and its modifications with non-standard modalities. Section 3 is devoted to the presentation of hypersequent calculi for the logics in question and the discussion of their meta-theoretical properties.

2 Semantics and axiom systems

Let us fix a modal language $\mathcal{L}_{\odot}$, where $\odot$ is an unary operation from the set $\{\Box, \Diamond, \rhd, \blacktriangleright, \circ, \bullet\}$ (these symbols stand for necessity, possibility, non-contingency, contingency, essence, and accident operators, respectively), with the alphabet $\langle \mathcal{P}, \odot, \neg, \vee, \wedge, \to, (,)\rangle$, where $\mathcal{P}$ is the set $\{p, q, r, p_1, \ldots\}$ of propositional variables. The set $\mathcal{F}_{\odot}$ of all $\mathcal{L}_{\odot}$-formulas is defined in a standard inductive way. We write $\mathcal{L}_{\odot\circledast}$, where $\odot$ and $\circledast$ are unary operations from the set $\{\Box, \Diamond, \rhd, \blacktriangleright, \circ, \bullet\}$ such that $\odot \neq \circledast$, for a bimodal language with both $\odot$ and $\circledast$ in its alphabet. Analogously, we write $\mathcal{F}_{\odot\circledast}$ for the set of all formulas of this bimodal language.

The logic S5 can be built in three languages: $\mathcal{L}_{\Box}$, $\mathcal{L}_{\Diamond}$, and $\mathcal{L}_{\Box\Diamond}$. We consider the latter variant. A pair $\langle W, V\rangle$ is an S5-model, if $W \neq \emptyset$ and V is a mapping from $W \times \mathcal{F}_{\Box\Diamond}$ to $\{1, 0\}$ such that it preserves classical conditions for truth-functional connectives and for any $A \in \mathcal{F}_{\Box\Diamond}$ and $x \in W$ we have:

- $V(\Box A, x) = 1$ iff $\forall_{y \in W} V(A, y) = 1$,
- $V(\Diamond A, x) = 1$ iff $\exists_{y \in W} V(A, y) = 1$.

The axiom system for S5 has all classical axioms, modus ponens, substitution rule, Gödel's rule (if $\vdash A$, then $\vdash \Box A$), and the subsequent modal axioms:

(1) $\Box(p \to q) \to (\Box p \to \Box q)$,
(2) $\Box p \to p$,
(3) $\Diamond p \to \Box\Diamond p$,
(4) $\Diamond p \leftrightarrow \neg\Box\neg p$.

A semantic condition for the non-contingency operator $\rhd$ is as follows:

- $V(\rhd A, x) = 1$ iff $\forall_{y \in W} V(A, y) = 1$ or $\forall_{y \in W} V(A, y) = 0$.

Following Montgomery and Routley [7] and Zolin [11], by the non-contingency version of S5 we mean the logic $\mathsf{S5}^{\rhd}$ which is the smallest set closed under modus ponens, substitution rule, Gödel's rule for $\rhd$ (if $\vdash A$, then $\vdash \rhd A$) and contaning all classical axioms as well as the subsequent modal ones:

(1) $p \to \big(\rhd(p \to q) \to (\rhd p \to \rhd q)\big)$, (3) $\rhd\rhd p$.
(2) $\rhd p \leftrightarrow \rhd\neg p$,

A semantic condition for the contingency operator is presented below:

- $V(\blacktriangleright A, x) = 1$ iff $\exists_{y\in W} V(A, y) = 1$ and $\exists_{y\in W} V(A, y) = 0$.

Let us present one of Montgomery and Routley's [7] axiom systems for $\mathsf{S5}^{\blacktriangleright}$. It is obtained from the one for $\mathsf{S5}^{\rhd}$ by changing $\rhd$ to $\neg\blacktriangleright$ in Gödel's rule and replacement of the axioms for $\rhd$ with the following ones:

(1) $p \to \big(\neg\blacktriangleright(p \to q) \to (\blacktriangleright q \to \blacktriangleright p)\big)$, (3) $\neg\blacktriangleright\blacktriangleright p$.
(2) $\blacktriangleright p \leftrightarrow \blacktriangleright\neg p$,

A semantic condition for the essence operator is as follows:

- $V(\circ A, x) = 1$ iff $V(A, x) = 0$ or $\forall_{y\in W} V(A, y) = 1$.

Axiomatization of $\mathsf{S5}^{\circ}$ was developed by Fan [2] and it has all classical axioms, modus ponens, substitution rule, and the following modal axioms and inference rule:

(1) $\circ\top$,
(2) $\neg p \to \circ p$,
(3) $(\circ p \wedge \circ q) \to \circ(p \wedge q)$,
(4) $p \to \circ(\circ\neg p \to p)$,
(5) $\neg\circ\neg p \to \circ(\circ\neg p \to p)$,
(6) if $\vdash A \to B$, then $\vdash (A \wedge \circ A) \to B$.

A semantic condition for the accident operator is as follows:

- $V(\bullet A, x) = 1$ iff $V(A, x) = 1$ and $\exists_{y\in W} V(A, y) = 0$.

Axiomatization of $\mathsf{S5}^{\bullet}$ can be obtained from Fan's axiomatization of $\mathsf{S5}^{\circ}$ due to equations $\bullet A = \neg\circ A$ and $\circ A = \neg\bullet A$. It has all classical axioms, modus ponens, substitution rule, and the following modal axioms and inference rule:

(1) $\neg\bullet\top$,
(2) $\bullet p \to p$,
(3) $\bullet(p \wedge q) \to (\bullet p \vee \bullet q)$,
(4) $\bullet(\neg p \to \bullet\neg p) \to \neg p$,
(5) $\bullet(\neg p \to \bullet\neg p) \to \neg\bullet\neg p$,
(6) $\vdash A \to B$ yields $\vdash (A \wedge \neg\bullet A) \to B$.

3 Hypersequent calculi

If Γ and Δ are finite multisets of formulas (of one of the languages considered in the paper), then we say that an ordered pair written as $\Gamma \Rightarrow \Delta$ is a sequent. By a hypersequent we mean a multiset of sequents written as $\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$. A sequent $\Gamma \Rightarrow \Delta$ is valid in an $\mathsf{S5}$-model $\langle W, V\rangle$ iff for any $x \in W$ it holds that $V(A, x) = 1$ (for any $A \in \Gamma$) implies $V(B, x) = 1$ (for some $B \in \Delta$). A hypersequent is valid in an $\mathsf{S5}$-model iff at least of one its components is valid in the same model. The notion of a proof a hypersequent calculus are understood in the standard way. If a hypersequent H is provable in a hypersequent calculus, we write $\vdash_{hs} H$ (while $\vdash$ we use for provability in axiom systems).

Let us introduce Restall's hypersequent calculus for $\mathsf{S5}$ [9]. The only axiom is as follows: (Ax) $A \Rightarrow A$. The structural rules are as follows:

$$(\text{EW}\Rightarrow)\ \frac{H}{A \Rightarrow \ \mid H} \qquad (\Rightarrow\text{EW})\ \frac{H}{\Rightarrow A \mid H}$$

$$(\text{IC}\Rightarrow)\ \frac{A, A, \Gamma \Rightarrow \Delta \mid H}{A, \Gamma \Rightarrow \Delta \mid H} \qquad (\Rightarrow\text{IC})\ \frac{\Gamma \Rightarrow \Delta, A, A \mid H}{\Gamma \Rightarrow \Delta, A \mid H}$$

$$(\text{Merge})\ \frac{\Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma \mid H}{\Gamma, \Pi \Rightarrow \Delta, \Sigma \mid H} \qquad (\text{Cut})\ \frac{\Gamma \Rightarrow \Delta, A \mid H \qquad A, \Pi \Rightarrow \Sigma \mid G}{\Gamma, \Pi \Rightarrow \Delta, \Sigma \mid H \mid G}$$

It's easy to observe that internal weaking rules as well as external contraction rules are derivable. Propositional logical rules are as follows:

$$(\neg \Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, A \mid H}{\neg A, \Gamma \Rightarrow \Delta \mid H} \qquad (\Rightarrow \neg)\ \frac{A, \Gamma \Rightarrow \Delta \mid H}{\Gamma \Rightarrow \Delta, \neg A \mid H}$$

$$(\vee \Rightarrow)\ \frac{A, \Gamma \Rightarrow \Delta \mid H \qquad B, \Gamma \Rightarrow \Delta \mid G}{A \vee B, \Gamma \Rightarrow \Delta \mid H \mid G} \qquad (\Rightarrow \vee)\ \frac{\Gamma \Rightarrow \Delta, A, B \mid H}{\Gamma \Rightarrow \Delta, A \vee B \mid H}$$

$$(\wedge \Rightarrow)\ \frac{A, B, \Gamma \Rightarrow \Delta \mid H}{A \wedge B, \Gamma \Rightarrow \Delta \mid H} \qquad (\Rightarrow \wedge)\ \frac{\Gamma \Rightarrow \Delta, A \mid H \qquad \Gamma \Rightarrow \Delta, B \mid G}{\Gamma \Rightarrow \Delta, A \wedge B \mid H \mid G}$$

$$(\rightarrow \Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, A \mid H \qquad B, \Pi \Rightarrow \Sigma \mid G}{A \rightarrow B, \Gamma, \Pi \Rightarrow \Delta, \Sigma \mid H \mid G} \qquad (\Rightarrow \rightarrow)\ \frac{A, \Gamma \Rightarrow \Delta, B \mid H}{\Gamma \Rightarrow \Delta, A \rightarrow B \mid H}$$

Modal logical rules are given below.

$$(\Box \Rightarrow)\ \frac{A, \Gamma \Rightarrow \Delta \mid H}{\Box A \Rightarrow \ \mid \Gamma \Rightarrow \Delta \mid H} \qquad (\Rightarrow \Box)\ \frac{\Rightarrow A \mid H}{\Rightarrow \Box A \mid H}$$

Although Restall himself did not consider the rules for $\Diamond$, they were suggested for his calculus in [5]:

$$(\Diamond \Rightarrow)\ \frac{A \Rightarrow \ \mid H}{\Diamond A \Rightarrow \ \mid H} \qquad (\Rightarrow \Diamond)\ \frac{\Gamma \Rightarrow \Delta, A \mid H}{\Gamma \Rightarrow \Delta \mid \Rightarrow \Diamond A \mid H}$$

The rules for non-standard modalities are given below.

$$(\triangleright \Rightarrow)\ \frac{A, \Gamma \Rightarrow \Delta \mid H \qquad \Pi \Rightarrow \Sigma, A \mid G}{\triangleright A \Rightarrow \ \mid \Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma \mid H \mid G} \qquad (\Rightarrow \triangleright)\ \frac{\Rightarrow A \mid A \Rightarrow \ \mid H}{\Rightarrow\ \triangleright A \mid H}$$

$$(\blacktriangleright \Rightarrow)\ \frac{\Rightarrow A \mid A \Rightarrow \ \mid H}{\blacktriangleright A \Rightarrow \ \mid H} \qquad (\Rightarrow \blacktriangleright)\ \frac{A, \Gamma \Rightarrow \Delta \mid H \qquad \Pi \Rightarrow \Sigma, A \mid G}{\Rightarrow \blacktriangleright A \mid \Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma \mid H \mid G}$$

$$(\circ \Rightarrow)\ \frac{A, \Gamma \Rightarrow \Delta \mid H \qquad \Pi \Rightarrow \Sigma, A \mid G}{\circ A, \Pi \Rightarrow \Sigma \mid \Gamma \Rightarrow \Delta \mid H \mid G} \qquad (\Rightarrow \circ)\ \frac{\Rightarrow A \mid A, \Gamma \Rightarrow \Delta \mid H}{\Gamma \Rightarrow \Delta, \circ A \mid H}$$

$$(\bullet \Rightarrow)\ \frac{\Rightarrow A \mid A, \Gamma \Rightarrow \Delta \mid H}{\bullet A, \Gamma \Rightarrow \Delta \mid H} \qquad (\Rightarrow \bullet)\ \frac{A, \Gamma \Rightarrow \Delta \mid H \qquad \Pi \Rightarrow \Sigma, A \mid G}{\Pi \Rightarrow \Sigma, \bullet A \mid \Gamma \Rightarrow \Delta \mid H \mid G}$$

Let $\odot \in \{\triangleright, \blacktriangleright, \circ, \bullet\}$. A hypersequent calculus for the logic $\mathsf{S5}^{\odot}$ is obtained from Restall's one for $\mathsf{S5}$ by the replacement of the rules for $\Box$ and $\Diamond$ with the rules for $\odot$.

Theorem 3.1 (Soundness) *Let* $\odot \in \{\triangleright, \blacktriangleright, \circ, \bullet\}$*. For every* $\mathcal{L}_{\odot}$*-formula* A*,* *it holds that* $\mathsf{S5}^{\odot} \vdash_{hs} A$ *implies* $\mathsf{S5}^{\odot} \models A$.

Theorem 3.2 (Equivalence) *Let* $\odot \in \{\rhd, \blacktriangleright, \circ, \bullet\}$. *For every* $\mathcal{L}_{\odot}$*-formula* A, *it holds that* $\mathsf{S5}^{\odot} \vdash A$ *implies* $\mathsf{S5}^{\odot} \vdash_{hs} A$.

As a consequence of Theorem 3.2 and the completeness of axiomatic systems for the logics in question, we obtain the the completeness result for our hypersequent calculi.

Theorem 3.3 (Completeness) *Let* $\odot \in \{\rhd, \blacktriangleright, \circ, \bullet\}$. *For every* $\mathcal{L}_{\odot}$*-formula* A, *it holds that* $\mathsf{S5}^{\odot} \models A$ *implies* $\mathsf{S5}^{\odot} \vdash_{hs} A$.

Theorem 3.4 (Cut elimination) *Let* $\odot \in \{\rhd, \blacktriangleright, \circ, \bullet\}$. *The rule* (Cut) *is eliminable in* $\mathsf{S5}^{\odot}$.

References

[1] Bednarska, K. and A. Indrzejczak, *Hypersequent calculi for S5: the methods of cut elimination*, Logic and Logical Philosophy **24** (2015), pp. 277–311.

[2] Fan, J., *Logics of essence and accident*, arXiv:1506.01872v1 (2015).

[3] Fine, K., *Essence and modality*, Philosophical Perspectives **8** (1994), pp. 1–16.

[4] Fine, K., *The logic of essence*, Journal of Philosophical Logic **24** (1995), pp. 241–273.

[5] Grigoriev, O. and Y. Petrukhin, *On a multilattice analogue of a hypersequent S5 calculus*, Logic and Logical Philosophy **28** (2019), pp. 683–730.

[6] Indrzejczak, A., *Two is enough – bisequent calculus for S5*, in: *Frontiers of Combining Systems. FroCoS 2019* (2019), pp. 277–294.

[7] Montgomery, H. and R. Routley, *Contingency and noncontingency bases for normal modal logics*, Logique et Analyse **9** (1966), pp. 318–328.

[8] Montgomery, H. and R. Routley, *Noncontingency axioms for S4 and S5*, Logique et Analyse **11** (1968), pp. 422–424.

[9] Restall, G., *Proofnets for S5: Sequents and circuits for modal logic*, in: *Logic Colloquium 2005* (2007), pp. 151–172.

[10] Small, C., *Reflections on Gödel's ontological argument*, in: *Klarheit in Religionsdingen: Aktuelle Beiträge zur Religionsphilosophie, vol. Band III of Grundlagenprobleme unserer Zeit* (2001), pp. 109–144.

[11] Zolin, E., *Sequential logic of arithmetical noncontingency*, Moscow University Mathematical Bulletin **56** (2001), pp. 43–48.

[12] Zolin, E., *Sequential reflexive logics with noncontingency operator*, Mathematical Notes **72** (2002), pp. 784–798.

Algebraic Semantics of Intuitionistic Inquisitive and Dependence Logic

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Abstract

We introduce an algebraic semantics for propositional inquisitive and dependence logic based on intuitionistic logic, introduced in [5] *via* team-semantics. We prove the equivalence of the two semantics by proving a duality result between the category finite Kripke frames and finite, well-connected, core-generated intuitionistic inquisitive algebras.

Keywords: Inquisitive Logic, Dependence Logic, Intuitionistic Logic, Algebraic Semantics.

1 Introduction

In this work we introduce an algebraic semantics for propositional inquisitive and dependence logic based on intuitionistic logic, and we show some possible applications of this novel semantic framework.

Dependence logic was introduced by Väänänen [14] as an extension of first-order logic with dependence atoms. In its standard formulation, dependence logic is defined *via* team semantics, originally introduced by Hodges in [8], which generalizes standard Tarski's semantics by teams, namely set of assignments which map first-order variables to elements of the domain. In its propositional version, a team is a set of valuations mapping propositional atoms to either 1 or 0. Propositional dependence logic has been studied in [15], while [16] considers several extensions of classical logic using team semantics. Intuitively, the dependence atom $=\!(\overline{p}, q)$ expresses the fact that the value of the variable q is uniquely determined by the values of the variables $\overline{p}$. The constancy atom $=\!(p)$ can then be seen as a special case of the dependency atom, saying that the value of a variable is constant in the underlying team.

On the other hand, inquisitive logic was formally developed by Ciardelli, Groenendijk and Roelofsen in a series of articles, most notably in [4,6], where

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they introduced the so-called "support semantics". Differently from dependence logic, inquisitive logic was developed hand-in-hand with inquisitive semantics – a linguistic framework that aims at providing a uniform formal characterisation of both questions and statements in natural languages. In particular, polar questions expressing "*whether p holds or not*" are represented by an operator $?p$ defined using the inquisitive disjunction as $?p := p \vee\!\!\!\vee \neg p$

It is known that inquisitive and dependence logic are closely related [3,15]. They are both extensions classical logic that adopt team semantics and they are also expressively equivalent as they are both complete w.r.t. all downward closed team properties. In [5] Ciardelli, Iemhoff and Yang have built on this similarity to introduce `InqI` – a version of inquisitive logic which is based on intuitionistic, rather than classical logic and which can be easily provided with a dependency operator.

In this work we we introduce an algebraic semantics for `InqI` based on the previous work on algebraic semantics of inquisitive logic [7,12,2]. Interestingly, similar work on intuitionistic inquisitive logic is currently being developed by Holliday [9] and Punčochář [11].

2 Team Semantics of `InqI`

We define the set $\mathcal{L}_{\mathtt{InqI}}$ of formulas of `InqI` inductively as:

$$\phi ::= p \mid \bot \mid =\!\!(p) \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \to \phi \mid \phi \vee\!\!\!\vee \phi;$$

where $p \in \mathtt{AT}$ is an arbitrary atomic formula. Negation is defined as $\neg\phi := \phi \to \bot$ and the dependency atom can be defined from the constancy atom as $=\!\!(\overline{p}, q) := (\bigwedge_{i \leq n} =\!\!(p_i)) \to =\!\!(q)$. If a formula α is defined in the restricted signature $\{\bot, \wedge, \to, \vee\}$, then we say that ϕ is *standard*. We use greek letters α, β, γ... to denote standard formulas.

The semantics of `InqI` is a version of team semantics over intuitionistic Kripke models. Fist, recall that an *intuitionistic Kripke frame* is a partial order $\mathfrak{F} = (W, R)$. An *intuitionistic Kripke model* is a pair $\mathfrak{M} = (\mathfrak{F}, V)$, where $\mathfrak{F}$ is an intuitionistic Kripke frame and $V : \mathtt{AT} \to Up(W)$ a valuation of atomic formulas over upsets of $\mathfrak{F}$. Notice that a world w in a model can be viewed as an assignment $w : \mathtt{AT} \to 2$ – hence we write $w(p) = 1$ if and only if $p \in V(w)$.

Definition 2.1 Let $\mathfrak{M} = (W, R, V)$ be an intuitionistic Kripke model. A *team* is a subset $t \subseteq W$. A team s is an *extension* of a team t if $s \subseteq R[t]$.

A team can thus be considered as a set of assignments. The *team semantics* of `InqI` is defined as follow.

Definition 2.2 Let $\mathfrak{M} = (W, R, V)$ be an intuitionistic Kripke model. The notion of a formula $\phi \in \mathcal{L}_{\mathtt{InqI}}$ being *true on a team* $t \subseteq W$ is defined as follows:

$$\begin{aligned} \mathfrak{M}, t \vDash p &\iff \forall w \in t\ (w(p) = 1) \\ \mathfrak{M}, t \vDash \bot &\iff t = \emptyset \\ \mathfrak{M}, t \vDash =\!\!(p) &\iff \forall w, v \in t\ (w(p) = v(p)) \end{aligned}$$

$$\begin{array}{rcl} \mathfrak{M}, t \vDash \psi \wedge \chi & \iff & \mathfrak{M}, t \vDash \psi \text{ and } \mathfrak{M}, t \vDash \chi \\ \mathfrak{M}, t \vDash \psi \vee \chi & \iff & \exists s, r \subseteq t \text{ such that } s \cup r = t, \mathfrak{M}, s \vDash \psi \text{ and } \mathfrak{M}, r \vDash \chi \\ \mathfrak{M}, t \vDash \psi \rightarrow \chi & \iff & \forall s \text{ (if } s \subseteq R[t] \text{ and } \mathfrak{M}, s \vDash \psi \text{ then } \mathfrak{M}, s \vDash \chi) \\ \mathfrak{M}, t \vDash \psi \veebar \chi & \iff & \mathfrak{M}, t \vDash \psi \text{ or } \mathfrak{M}, t \vDash \chi. \end{array}$$

We then write $\mathfrak{M} \vDash \phi$ if $\mathfrak{M}, t \vDash \phi$ for all $t \subseteq W$ and $\mathfrak{F} \vDash \phi$ if $(\mathfrak{F}, V) \vDash \phi$ for all valuations V. The logic `InqI` is then defined as follow:

$$\texttt{InqI} := \{\phi \in \mathcal{L}_{\texttt{InqI}} : \mathfrak{F} \vDash \phi \text{ where } \mathfrak{F} \text{ is any intuitionistic Kripke frame}\}$$

We recall some important properties of such semantics [5, Prop. 3.15]. Recall that a formula ϕ is *flat* (or *truthconditional*) if for any model $\mathfrak{M}$ and team t, we have that $\mathfrak{M}, t \vDash \phi \iff \mathfrak{M}, w \vDash \phi$ for all $w \in t$.

Proposition 2.3

- *Persistency: if $\mathfrak{M}, t \vDash \phi$ and $s \subseteq R[t]$, then $M, s \vDash \phi$.*
- *Empty Team Property: $\mathfrak{M}, \emptyset \vDash \phi$, for all $\phi \in \mathcal{L}_{\texttt{InqI}}$.*
- *ϕ is flat if and only if there is a standard formula α such that $\phi \equiv \alpha$.*

Finally, let us notice what are the inquisitive and the dependency features of the logic defined above. Inquisitive logic is usually introduced over the signature $\{\bot, \wedge, \veebar, \rightarrow\}$, while the constancy atom comes from dependence logic. However, it is easy to check that $=\!\!(p) \equiv p \veebar \neg p$, so one could also decide not to take $=\!\!(\cdot)$ as a primitive symbol. On the other hand, note that the "intuitionistic" disjunction $\vee$ is not that given by the intuitionistic core logic, as we allow it to occur also in non-standard formulas. In fact, it is obtained by lifting the intuitionistic disjunction of standard formulas to the entire logic. This is known as *teamification* [10] in team-based logics. We believe the algebraic semantics of the next sections shall give new light to this phenomenon.

3 Algebraic Semantics for `InqI`

We shall now develop an alternative algebraic semantics for the system `InqI` defined in the previous section.

Definition 3.1 [Intuitionistic Inquisitive Algebra] An *intuitionistic inquisitive algebra* (or `InqI`-algebra) H is a tuple $(H, \veebar, \vee, \wedge, \rightarrow, 0, H_0)$, where:

- $(H, \veebar, \wedge, \rightarrow, 0)$ and $(H_0, \vee, \wedge, \rightarrow, 0)$ are Heyting algebras;
- $H_0 = \{\alpha \in H : \forall x, y[\alpha \rightarrow (x \veebar y) = (\alpha \rightarrow x) \veebar (\alpha \rightarrow y)]\}$;
- For all $x, y, z \in H$, the following equality hold:

$$(*) \quad x \vee (y \veebar z) = (x \vee y) \veebar (x \vee z).$$

And we then define the constancy atom as a partial operation $=\!\!(p) := p \veebar \neg p$. Clearly $(H_0, \vee, \wedge, \rightarrow, 0)$ is a subalgebra of $(H, \veebar, \wedge, \rightarrow, 0)$ w.r.t. the reduct $\{\wedge, \rightarrow, 0\}$. We often refer to the algebra H_0 as the *core* of the algebra H. Since negation is defined as $\neg x := x \to 0$, these two algebras also agree on their negation. A homomorphism between intuitionistic inquisitive algebras is any map $h : H \to H'$ such that $h(x \odot y) = h(x) \odot h(y)$ for $\odot \in \{\wedge, \vee\!\!\!\vee, \vee, \to, \vee, 0\}$. In general, if H is a Heyting algebra and $K \subseteq H$, then we denote by $\langle K \rangle$ the subalgebra of H generated by K. An `InqI`-algebra H is called *core-generated* if $H = \langle H_0 \rangle$.

Definition 3.2 An *intuitionistic inquisitive algebraic model* is a pair $M = (H, \mu)$ such that H is a `InqI`-algebra and $\mu : \mathtt{AT} \to H_0$ a *core valuation*, i.e. μ assigns atomic formulas in `AT` to elements in the core H_0.

The interpretation of an arbitrary formula $\phi \in \mathcal{L}_{\mathtt{InqI}}$ in an algebraic model $M = (H, \mu)$ is then defined as follows.

Definition 3.3 Given an intuitionistic inquisitive algebraic model M and a formula $\phi \in \mathcal{L}_{\mathtt{InqI}}$, its *interpretation* $[\![\phi]\!]^M$ is defined as follows:

$$\begin{aligned} [\![p]\!]^M &= \mu(p) & [\![\bot]\!]^M &= 0 \\ [\![\phi \wedge \psi]\!]^M &= [\![\phi]\!]^M \wedge [\![\psi]\!]^M & [\![\phi \vee \psi]\!]^M &= [\![\psi]\!]^M \vee [\![\chi]\!]^M \\ [\![\phi \to \psi]\!]^M &= [\![\phi]\!]^M \to [\![\psi]\!]^M & [\![\phi \vee\!\!\!\vee \psi]\!]^M &= [\![\phi]\!]^M \vee\!\!\!\vee [\![\psi]\!]^M \end{aligned}$$

We write $M \vDash^0 \phi$ if $[\![\phi]\!]^M = 1$. We say that ϕ is *valid in* H and write $H \vDash^0 \phi$ if ϕ is true in every model $M = (H, \mu)$ over H. Finally, we say that ϕ is an *algebraic validity* of `InqI` if it is true in all intuitionistic inquisitive models.

Proposition 3.4 (Normal Form) *Let H be a intuitionistic inquisitive algebra and $x \in \langle H_0 \rangle$, then $x = \bigvee\!\!\!\bigvee_{i \leq n} a_i$, for some $a_0, ...a_n \in H_0$.*

Theorem 3.5 *Let $\phi \in \mathcal{L}_{\mathtt{InqI}}$, then $H \nvDash^0 \phi$ entails $\langle H_0 \rangle \nvDash^0 \phi$.*

Finally, we can prove a characterisation of core-generated, well-connected `InqI`-algebras. Recall that, if H is a Heyting algebra, H is well-connected if $x \vee y = 1$ entails thay $x = 1$ or $y = 1$. Also, recall that $x \in H$ is *join irreducible* if $x = y \vee z$ entails that $x = y$ or $x = z$.

Theorem 3.6 *Suppose H is a finite, core-generated and well-connected Heyting algebra, then $\alpha \in H_0$ if and only if α is join-irreducible.*

4 Equivalence of Team and Algebraic Semantics

To prove the equivalence of team and algebraic semantics we shall first prove a categorical equivalence relating Kripke frames and intuitionistic inquisitive algebras. Let **FinKF** be the category of finite intuitionistic Kripke frames and p-morphisms and **FIIA** the category of finite, well-connected, core-generated, `InqI`-algebras and `InqI`-homomorphisms.

We sketch the proof of the equivalence **FinKF** $\cong$ **FIIA**. First, we describe how, given a intuitionistic Kripke frame, we can obtain a finite, core-generated intuitionistic inquisitive algebra:

$$\mathfrak{F} \longmapsto Up(\mathfrak{F}) \longmapsto Dw^+(Up(\mathfrak{F}))$$

Given an intuitionistic Kripke frame $\mathfrak{F} = (W, R)$, we first consider the algebra of its R-upsets $(Up(\mathfrak{F}), \cup, \cap, \emptyset)$. Since this is clearly a bounded distributive lattice, it is a Heyting algebra as well. Then, for the same argument, $Dw^+(Up(\mathfrak{F}))$, the set of all nonempty downsets of $Up(\mathfrak{F})$ ordered by the subset relation $\subseteq$ also forms a Heyting algebra. Notice that upsets are taken with respect to the relation R of the Kripke frame $\mathfrak{F}$, while downsets are here downward closed subsets over the algebra $(Up(\mathfrak{F}), \cup, \cap, \emptyset)$. Now, let F, G be two functors F : **FinKF** $\to$ **FIIA**, G : **FIIA** $\to$ **FinKF** such that $F : \mathfrak{F} \mapsto Dw^+(Up(\mathfrak{F}))$ and $G : H \mapsto \mathcal{PF}(H)$. Together with a result from Raney [13] – which allows us to represent Heyting algebras generated by their join-irreducible elements as algebras of downsets of such elements – it follows by Esakia duality that F and G describe an equivalence of categories, namely **FinKF** $\cong$ **FIIA**.

Now, suppose $\mathfrak{M} = (\mathfrak{F}, V)$ is a finite Kripke frame and let $H_{\mathfrak{F}}$ be $F(\mathfrak{F})$. To obtain an intuitionistic inquisitive model corresponding to M, define the core valuation $\mu(p) = \wp(V(p))$. One can then prove the following theorem.

Proposition 4.1 *Let $\mathfrak{M} = (\mathfrak{F}, V)$ be a finite Kripke frame and $M = (H_{\mathfrak{F}}, \mu)$, then $\mathfrak{M} \vDash \phi$ if and only if $M \vDash^0 \phi$.*

Finally, we obtain as a result the algebraic completeness of the logic `InqI`.

Theorem 4.2 (Equivalence of Team and Algebraic Semantics) *For any $\phi \in \mathcal{L}_{\mathtt{InqI}}$, ϕ is valid in all intuitionitic Kripke frames if and only if it is valid in all intuitionistic inquisitive algebras.*

5 Relation to Existing Works and Generalisations

In [11], Punčochář has introduced an algebraic semantics for intuitionistic inquisitive logic which is very similar to the one considered in this article. In particular, inquisitive Heyting algebras are introduced as algebras of antichains over bounded implicative meet semilattice. However, there are two important points worth stressing. Firstly, we have included two disjunctions in our signature, the tensor disjunction and the inquisitive disjunction. One can also "forget" the tensor disjunction and require $(H_0, \wedge, \to, 0)$ to be a bounded implicative meet semilattice. It is then clear how our approach turns out to be complementary to that of [11]. In particular, we expect the class of inquisitive algebras defined in [11] to result as a class of representatives for our corresponding class defined in more equational terms.

Secondly, an important aspect of such algebraic semantics, is that it allows us to consider some natural generalisations of this logic. In particular, it is very natural to consider intuitionistic inquisitive algebras whose core is the algebra of some intermediate logic. We then say that H is an *L-inquisitive algebra* if $H \vDash^0 L$, where L is any intermediate logic in the standard signature $\{\top, \bot, \wedge, \vee, \to\}$. Clearly, if $H \vDash^0$ `CPC`, we then have that H_0 is a Boolean Algebra, so H is a model of `InqB` and it indeed coincides with the algebraic semantics for standard inquisitive logic considered in [1,12,2].

Finally, a further direction is to develop an algebraic semantics for modal inquisitive and dependence logic. This aspect is particularly interesting as it relates to the translations between intuitionistic and modal inquisitive logic described in [5]. It is an interesting open problem to characterise in algebraic terms the translations between intermediate and modal inquisitive and dependence logics.

References

[1] Bezhanishvili, N., G. Grilletti and W. H. Holliday, *Algebraic and topological semantics for inquisitive logic via choice-free duality*, in: *Logic, Language, Information, and Computation. WoLLIC 2019.*, Springer, 2019 pp. 35–52.

[2] Bezhanishvili, N., G. Grilletti and D. E. Quadrellaro, *An algebraic approach to inquisitive and dna-logics* ILLC PP-2020-10.

[3] Ciardelli, I., "Dependency as Question Entailment," Springer International Publishing, Cham, 2016 pp. 129–181.

[4] Ciardelli, I., J. Groenendijk and F. Roelofsen, *Attention!* Might *in inquisitive semantics*, in: *Proceedings of Semantics and Linguistic Theory*, 2009 .

[5] Ciardelli, I., R. Iemhoff and F. Yang, *Questions and dependency in intuitionistic logic*, Notre Dame J. Formal Logic **61** (2020), pp. 75–115.

[6] Ciardelli, I. and F. Roelofsen, *Inquisitive logic*, Journal of Philosophical Logic **40** (2011), pp. 55–94.

[7] Grilletti, G. and D. E. Quadrellaro, *Lattices of intermediate theories via ruitenburg's theorem.*, arXiv: Logic (2020).

[8] Hodges, W., *Compositional semantics for a language of imperfect information*, Logic Journal of the IGPL **5** (1997), pp. 539–563.

[9] Holliday, W., *Inquisitive intuitionistic logic*, in: *Proceedings of Advances in Modal Logic* (2020), forthcoming.

[10] Lück, M., "Team Logic; Axioms, Expressiveness, Complexity," 2019, phD thesis, Gottfried Wilhelm Leibniz Universität Hannover.

[11] Punčochář, V., *Inquisitive heyting algebras* (2020), unpublished Manuscript.

[12] Quadrellaro, D. E., *Lattices of* DNA*-logics and algebraic semantics of inquisitive logic* (2019), msc Thesis, University of Amsterdam.

[13] Raney, G. N., *Completely distributive complete lattices*, Proceedings of the American Mathematical Society **3** (1952), pp. 677–680.

[14] Väänänen, J., "Dependence Logic: A New Approach to Independence Friendly Logic," Cambridge University Press, 2007.

[15] Yang, F. and J. Väänänen, *Propositional logics of dependence*, Annals of Pure and Applied Logic **167** (2016), pp. 557–589.

[16] Yang, F. and J. Väänänen, *Propositional team logics*, Annals of Pure and Applied Logic **168** (2017), pp. 1406–1441.

Canonical extensions for the Distributive Full Lambek Calculus with Modal Operators

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Abstract

In this paper, we investigate logic of bounded distributive residuated lattices with modal operators $\Box$ and $\Diamond$. We introduce relational semantics for such substructural modal logics. We prove that any canonical logic is Kripke complete using discrete duality and canonical extensions. See this preprint [7] to have more details.

Keywords: The Lambek calculus, canonical extensions, residuated lattices

1 Introduction

Substructural logic is logic lacking some of the well-known structural rules such as contraction, weakening, or exchange. Algebraically, substructural logics represent ordered residuated algebras [6]. In this talk, we consider the distributive version of the full Lambek calculus extended with normal modal operators $\Box$ and $\Diamond$. We introduce ternary Kripke frames, relational structures for the distributive Lambek calculus extended with binary modal relations. We establish a discrete duality between such Kripke frames and perfect distributive residuated modal algebras developing an approach proposed in [5]. We examine canonical extensions for those algebras applying techniques provided in [2] [4] to show that any canonical substructural distributive modal logic is Kripke complete.

2 The distributive Lambek calculus with modal operators

We introduce the distributive full Lambek calculus enriched with modal operators. We represent such logics with pairs that have the form $\varphi \vdash \psi$. φ and ψ are formulas generated by the grammar of the full Lambek calculus with $\Box$ and $\Diamond$.

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Definition 2.1 A substructural normal distributive modal logic is a set of pairs Λ including the following axioms and inference rules:

- $\bot \vdash p,\ p \vdash \top$
- $p \vdash p$
- $p_i \vdash p_1 \vee p_2, i = 1, 2$
- $p_1 \wedge p_2 \vdash p_i, i = 1, 2$
- $p \wedge (q \vee r) \vdash (p \wedge q) \vee (p \wedge r)$
- $(p \bullet q) \bullet r \dashv\vdash p \bullet (q \bullet r)$
- From $\varphi \vdash \psi$ and $\psi \vdash \theta$ infer $\varphi \vdash \theta$
- From $\varphi \vdash \psi$ and $\theta \vdash \psi$ infer $\varphi \vee \theta \vdash \psi$
- From $\varphi \bullet \theta \vdash \psi$ infer $\theta \vdash \varphi \setminus \psi$ and vice versa
- From $\varphi \vdash \psi$ infer $\Box\varphi \vdash \Box\psi$
- $p \bullet \mathbf{1} \dashv\vdash \mathbf{1} \bullet p \dashv\vdash p$
- $\Diamond(p \vee q) \vdash \Diamond p \vee \Diamond q$
- $\Diamond\bot \vdash \bot$
- $\Box p \wedge \Box q \vdash \Box(p \wedge q)$
- $\top \vdash \Box\top$
- $\Box p \bullet \Box q \vdash \Box(p \bullet q)$
- From $\varphi(p) \vdash \psi(p)$ infer $\varphi[p := \gamma] \vdash \psi[p := \gamma]$
- From $\varphi \vdash \psi$ and $\varphi \vdash \theta$ infer $\varphi \vdash \psi \wedge \theta$
- From $\theta \vdash \varphi \setminus \psi$ infer $\varphi \bullet \theta \vdash \psi$ and vice versa
- From $\varphi \vdash \psi$ infer $\Diamond\varphi \vdash \Diamond\psi$

Substructural normal distributive modal logic extends distributive normal modal logic (see [5]) with residuals, product, and the axiom connecting $\Box$ and $\bullet$. We define a ternary Kripke frame with the additional binary modal relations. Product and residuals have the ternary semantics as in, e.g., [1].

Definition 2.2 A modal ternary Kripke frame is a structure $\mathcal{F} = \langle W, \leq, R, R_\Box, R_\Diamond, \mathcal{O}\rangle$, where $\langle W, \leq, \rangle$ is a partial order, R is a ternary relation on W, $R_\Box, R_\Diamond$ are binary relations on W, and $\mathcal{O} \subseteq W$ such that for all $\forall u, v, w, u', v', w' \in W$:

(i) $Ruvw \,\&\, wR_\Box w' \Rightarrow \exists x, y \in W\ \ Rxyw' \,\&\, uR_\Box x \,\&\, vR_\Box y$.

(ii) $\exists x \in W(Ruwx \,\&\, Rxu'v') \Leftrightarrow \exists y \in W(Rwu'y \,\&\, Ruyv')$.

(iii) $Ruvw \,\&\, u' \leq u \Rightarrow Ru'vw$, $Ruvw \,\&\, v' \leq v \Rightarrow Ruv'w$, $Ruvw \,\&\, w \leq w' \Rightarrow Ruvw'$.

(iv) $\forall o \in \mathcal{O}\ Rvow \Leftrightarrow Rovw$, $v \leq w \Leftrightarrow \exists o \in \mathcal{O} Rvow$, and $\mathcal{O}$ is upwardly closed.

(v) $u \leq v \,\&\, vR_\Box w \Rightarrow uR_\Box w$ and $u \leq v \,\&\, uR_\Diamond w \Rightarrow vR_\Diamond w$.

Definition 2.3 Let $\mathcal{F} = \langle W, \leq, R, R_\Box, R_\Diamond, \mathcal{O}\rangle$ be a modal ternary Kripke frame, a Kripke model is a pair $\mathcal{M} = \langle \mathcal{F}, \vartheta\rangle$, where $\vartheta : \mathrm{PV} \to \mathrm{Up}(W, \leq)$.

(i) $\mathcal{M}, w \models p \Leftrightarrow w \in \vartheta(p)$.

(ii) $\mathcal{M}, w \models \top$; $\mathcal{M}, w \not\models \bot$; $\mathcal{M}, w \models \mathbf{1} \Leftrightarrow w \in \mathcal{O}$.

(iii) $\mathcal{M}, w \models \varphi \bullet \psi \Leftrightarrow \exists u, v \in W\ Ruvw \,\&\, \mathcal{M}, u \models \varphi \,\&\, \mathcal{M}, v \models \psi$.

(iv) $\mathcal{M}, w \models \varphi \setminus \psi \Leftrightarrow \forall u, v \in W\ Ruwv \,\&\, \mathcal{M}, u \models \varphi$ implies $\mathcal{M}, v \models \psi$.

(v) $\mathcal{M}, w \models \psi / \varphi \Leftrightarrow \forall u, v \in W\ Ruwv \,\&\, \mathcal{M}, u \models \varphi$ implies $\mathcal{M}, v \models \psi$.

(vi) $\mathcal{M}, w \models \varphi \wedge \psi \Leftrightarrow \mathcal{M}, w \models \varphi \,\&\, \mathcal{M}, w \models \psi$.

(vii) $\mathcal{M}, w \models \varphi \vee \psi \Leftrightarrow \mathcal{M}, w \models \varphi$ or $\mathcal{M}, w \models \psi$.

(viii) $\mathcal{M}, w \models \Box\varphi \Leftrightarrow \forall v \in R_{\Box}(w)\ \mathcal{M}, v \models \varphi$.

(ix) $\mathcal{M}, w \models \Diamond\varphi \Leftrightarrow \exists v \in R_{\Diamond}(w)\ \mathcal{M}, v \models \varphi$.

(x) $\mathcal{M}, w \models \varphi \vdash \psi \Leftrightarrow \mathcal{M}, w \models \varphi \Rightarrow \mathcal{M}, w \models \psi$.

Let $\mathcal{F}$ be a modal ternary Kripke frame and $\varphi \vdash \psi$ a pair of formulas, $\mathcal{F} \models \varphi \vdash \psi$ iff for each valuation ϑ $\langle \mathcal{F}, \vartheta \rangle \models \varphi \vdash \psi$. $\mathrm{Log}(\mathcal{F}) = \{\varphi \vdash \psi \mid \mathcal{F} \models \varphi \vdash \psi\}$. Let $\mathbb{F}$ be a class of modal ternary Kripke frames, then $\mathrm{Log}(\mathbb{F}) = \cap_{\mathcal{F} \in \mathbb{F}} \mathrm{Log}(\mathcal{F})$. Let Λ be a substructural normal modal logic, $\mathrm{Frames}(\mathcal{L}) = \{\mathcal{F} | \mathcal{F} \models \mathcal{L}\}$ and $\mathcal{L}$ is *complete* iff $\mathcal{L} = \mathrm{Log}(\mathrm{Frames}(\mathcal{L}))$. By $\mathbf{L_K}$, we mean the minimal substructural distributive normal modal logic, the smallest set of pairs including the axioms above and is closed under the required inference rules.

Theorem 2.4 *Let* $\mathbb{F}$ *be a class of modal ternary Kripke frames, then* $\mathrm{Log}(\mathbb{F})$ *is a substructural distributive normal modal logic.*

3 Residuated distributive modal algerbas

In this section, we study algebraic semantics and canonical extensions for substructural distributive modal logic.

Definition 3.1 A residuated lattice is an algebra $\mathcal{R} = \langle \mathcal{L}, \cdot, \backslash, /, \varepsilon \rangle$, where $\mathcal{L}$ is a bounded lattice, $\cdot$ is a binary associative monotone operation, ε is a multiplicative identity, and the following equivalences hold for all $a, b, c \in \mathcal{L}$:

$$b \leq a \setminus c \Leftrightarrow a \cdot b \leq c \Leftrightarrow a \leq c/b$$

Definition 3.2 Let R be a bounded distributive residuated lattice, a residuated distributive modal algebra (RDMA) is an algebra $\mathcal{M} = \langle \mathcal{R}, \Box, \Diamond \rangle$ such that $\Box$ preserves finite suprema, $\Diamond$ preserves finite infima, and for each $a, b \in \mathcal{R}$ one has $\Box a \cdot \Box b \leq \Box(a \cdot b)$.

Definition 3.3 Let Λ be a substructural normal modal logic, $\mathcal{V}_\Lambda$ is a variety of RDMAs defined by the set of inequations $\{\varphi \leq \psi \mid \Lambda \vdash \varphi \vdash \psi\}$.

Theorem 3.4 *Let* Λ *be a substructural normal modal logic, then there exists an RDMA* $\mathcal{R}_\Lambda$ *such that* $\varphi \vdash \psi \in \Lambda$ *iff* $\mathcal{R}_\Lambda \models \varphi \leq \psi$.

We define a completely distributive residuated perfect lattice as a distributive version of a residuated perfect one defined in [2].

Definition 3.5 A distributive residuated lattice $\mathcal{L} = \langle L, \bigvee, \bigwedge, \cdot, \backslash, /, \varepsilon \rangle$ is called perfect distributive residuated lattice, if:

- Its lattice reduct is completely distributive.
- $\cdot$, $\backslash$, and $/$ are binary operations on L such that $/$ and $\backslash$ right and left residuals of $\cdot$, repsectively; $\cdot$ is a complete operator on $\mathcal{L}$, and $/ : \mathcal{L} \times \mathcal{L}^\delta \to \mathcal{L}$, $\backslash : \mathcal{L}^\delta \times \mathcal{L} \to \mathcal{L}$ are complete dual operators, where $\mathcal{L}^\delta$ is the dual of $\mathcal{L}$.

We formulate canonical extensions for bounded distributive lattices with a residuated family in the fashion of [3]. We piggyback canonical extensions for bounded distributive lattice expansions. We refer to this paper [4] and omit the abstract definitions. We only recall that a canonical extension (a dense and compact completion) of a bounded distributive lattice is completely distributive.

Lemma 3.6 *Let $\mathcal{L} = \langle L, \cdot, \backslash, /, \varepsilon\rangle$ be a bounded distributive residuated lattice, then $\mathcal{L}^\sigma = \langle L^\sigma, \cdot^\sigma, \backslash^\pi, /^\pi, \varepsilon\rangle$ is a perfect distributive residuated lattice.*

We just define $\cdot^\sigma$, $\backslash^\pi$, and $/^\pi$ explicitly instead of providing a proof that mostly repeats a construction from the paper by Gehrke [3].

Let a, a' be filter elements of $\mathcal{L}^\sigma$ and b an ideal one:

(i) $a \backslash^\pi b = \bigvee\{x \backslash y \mid a \leq x \in \mathcal{L} \ni y \leq b\}$ and similarly for the right residual.

(ii) $a \cdot^\sigma a' = \bigwedge\{x \cdot x' \mid a \leq x \in \mathcal{L} \,\&\, a \leq x' \in \mathcal{L}\}$.

Let $a, b \in \mathcal{L}^\sigma$, then.

(i) $a \cdot^\sigma b = \bigvee\{x \cdot^\sigma y \mid a \geq x \,\&\, b \geq y\}$, where x, y are filter elements

(ii) $a \backslash^\pi b = \bigwedge\{x \backslash^\pi y \mid a \geq x \,\&\, b \leq y\}$, where x is a filter element and y is an ideal one. The $b/^\pi a$ case is similar to the current one.

We concretise the construction establishing the discrete duality between perfect residuated lattices and perfect posets with ternary relation (see [2]) in a distributive setting.

Let $\mathcal{L}$ be a perfect distributive residuated lattice. We define a relation R on completely join-irreducible elements as $Rabc \Leftrightarrow a \cdot b \leq c$ and put $\mathcal{O} = \uparrow \varepsilon$, where ε is a multiplicative identity. The structure $\mathcal{L}_+ = \langle \mathcal{J}^\infty(\mathcal{L}), \leq, R, \mathcal{O}\rangle$ is the *dual ternary Kripke frame* of a perfect distributive residuated lattice $\mathcal{L}$.

Let $\langle W, \leq\rangle$ be a poset and $R \subseteq W^3$, $\mathcal{O}$ with the conditions (ii)-(vi) from Definition 2.2. Let us define the operations on $\mathrm{Up}(W, \leq)$ as follows:

- $A \backslash B = \{w \in W \mid \forall u, v \in W \; Ruwv \,\&\, u \in A \Rightarrow v \in B\}$
- $B/A = \{w \in W \mid \forall u, v \in W \; Rwuv \; v \in A \Rightarrow v \in B\}$
- $A \cdot B = \{w \in W \mid \exists u, v \in W \; Ruvw \,\&\, u \in A \,\&\, v \in B\}$

Let us call such a poset with a relation a *ternary Kripke frame.*

Theorem 3.7

(i) *Let $\mathcal{R}$ be a perfect distributive residuated lattice, then $\mathcal{R} \cong (\mathcal{R}_+)^+$.*

(ii) *Let $\mathcal{F}$ be a ternary Kripke frame, then $\mathcal{F} \cong (\mathcal{F}^+)_+$.*

Definition 3.8 Let $\mathcal{L}$ be a perfect distributive residuated lattice and $\Box, \Diamond$ unary operators on $\mathcal{L}$, then $\mathcal{M} = \langle \mathcal{L}, \Box, \Diamond\rangle$ is called a perfect distributive residuated modal algebra, if $\Box$ is completely multiplicative, $\Diamond$ is completely additive, and for each $a, b \in \mathcal{L}$ the inequation $\Box a \cdot \Box b \leq \Box(a \cdot b)$ holds.

Lemma 3.9 *Let $\mathcal{R}$ be a distributive residuated lattice and $\mathcal{M} = \langle \mathcal{R}, \Box, \Diamond\rangle$ an RDMA, then $\mathcal{M}^\sigma = \langle \mathcal{R}^\sigma, \Box^\sigma, \Diamond^\sigma\rangle$ is a perfect DRMA. That is, the variety of all RDMAs is canonical.*

Proof. Let a, b be filter elements. Note that $\Box^\sigma a \cdot^\sigma \Box^\sigma b = \bigwedge\{\Box x \cdot \Box y \mid a \leq x \in \mathcal{M}, b \leq y \in \mathcal{M}\}$ that follows from the definition of a filter element, the fact that $\Box^\sigma$ preserves all infima and $\cdot^\sigma$ is an order-preserving operation. One has:

$\Box^\sigma a \cdot^\sigma \Box^\sigma b =$
$\bigwedge\{\Box x \cdot \Box y \mid a \leq x \in \mathcal{L}, \leq x \in \mathcal{L}\} \leq \bigwedge\{\Box(x \cdot y) \mid a \leq x \in \mathcal{L} \,\&\, b \leq x \in \mathcal{L}\} =$
$\bigwedge \Box^\sigma\{(x \cdot y) \mid a \leq x \in \mathcal{L} \,\&\, b \leq x \in \mathcal{L}\} =$
$\Box^\sigma \bigwedge\{(x \cdot y) \mid a \leq x \in \mathcal{L} \,\&\, b \leq x \in \mathcal{L}\} = \Box^\sigma(a \cdot^\sigma b)$

Let $a, b \in \mathcal{L}^\sigma$, then
$\Box^\sigma a \cdot^\sigma \Box b = \bigvee\{\Box^\sigma x \cdot^\sigma \Box^\sigma y \mid a \geq x \in \mathcal{C}(\mathcal{L}^\sigma) \,\&\, b \geq y \in \mathcal{C}(\mathcal{L}^\sigma)\} \leq$
$\bigvee\{\Box^\sigma(x \cdot^\sigma y) \mid a \geq x \in \mathcal{C}(\mathcal{L}^\sigma) \,\&\, b \geq y \in \mathcal{C}(\mathcal{L}^\sigma)\} \leq$ □
$\Box^\sigma \bigvee\{x \cdot^\sigma y \mid a \geq x \in \mathcal{C}(\mathcal{L}^\sigma) \,\&\, b \geq y \in \mathcal{C}(\mathcal{L}^\sigma)\} = \Box^\sigma(a \cdot b)$

Definition 3.10 A substructural normal modal logic $\mathcal{L}$ is called canonical if $\mathcal{V}_{\mathcal{L}}$ is closed under canonical extensions.

Now we describe a discrete duality for RDMAs explicitly. The complex algebra of a modal ternary Kripke frame $\mathcal{F} = \langle W, \leq, R, R_\Box, R_\Diamond, \mathcal{O}\rangle$ is the complex algebra of the underlying residuated frame $\mathcal{F}^+$ with the modal operators defined as $[R_\Box]A = \{u \in W \mid \forall w\ (uR_\Box w \Rightarrow w \in A)\}$ and $\langle R_\Diamond\rangle = \{u \in W \mid \exists w\ (uR_{\Diamond w} \,\&\, w \in A)\}$. Here A is an upwardly closed subset. These operations are well-defined. The dual modal ternary frame of a perfect RDMA $\mathcal{M} = \langle M, \bigvee, \bigwedge, \Box, \Diamond, \cdot, \backslash, /, \varepsilon\rangle$ is the dual frame $\mathcal{M}_+$ of an underlying perfect distributive residuated lattice with binary relations on completely join irreducible elements. We define these relations as $aR_\Box b \Leftrightarrow \Box\kappa(a) \leq \kappa(b)$ and $aR_\Diamond b \Leftrightarrow a \leq \Diamond b$ [2].

Theorem 3.11

(i) *Let $\mathcal{F}$ be a modal ternary Kripke frame, then $\mathcal{F} \cong (\mathcal{F}^+)_+$.*
(ii) *Let $\mathcal{M}$ be a perfect DRMA, then $\mathcal{M} \cong (\mathcal{M}_+)^+$.*

Theorem 3.12 *Let $\mathcal{L}$ be a canonical substructural distributive modal logic, then $\mathcal{L}$ is Kripke complete (and $\mathbf{L_K}$ as well).*

References

[1] Došen, K., *A brief survey of frames for the lambek calculus*, Mathematical Logic Quarterly **38** (1992), pp. 179–187.

[2] Dunn, J. M., M. Gehrke and A. Palmigiano, *Canonical extensions and relational completeness of some substructural logics*, J. Symbolic Logic **70** (2005), pp. 713–740.

[3] Gehrke, M., *Topological duality and algebraic completions*, Chapter to appear in Hiroakira Ono on Residuated Lattices and Substructural Logics, Outstanding Contributions Series, Springer (2015).

[4] Gehrke, M. and B. Jónsson, *Bounded distributive lattice expansions*, Mathematica Scandinavica **94** (2004), pp. 13–45.

[5] Gehrke, M., H. Nagahashi and Y. Venema, *A sahlqvist theorem for distributive modal logic*, Annals of Pure and Applied Logic **131** (2005), pp. 65 – 102.

[6] Ono, H., *Substructural logics and residuated lattices—an introduction*, in: *Trends in logic*, Springer, 2003 pp. 193–228.

[7] Rogozin, D., *The distributive full lambek calculus with modal operators*, arXiv preprint arXiv:2003.09975 (2020).

[2] Here, κ is an order isomorphism between completely join- and meet-irreducible elements.

Generalised Veltman Semantics in Agda

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Abstract

In this extended abstract we compute some rather involved frame conditions w.r.t. Generalised Veltman Semantics for principles of interpretability logic. All proofs have been formalised in Agda and we briefly comment on this formalisation.

Keywords: Interpretability, Provability logic, Veltman semantics, Agda.

1 Preliminaries

Interpretability logics aim to capture the provably structural behavior of formalised interpretability in the same sense as provability logics do for formalised provability. While any reasonable theory has the same provability logic this is not the case for interpretability, and reasonable finitely axiomatised theories have a different interpretability logic than theories with full induction. A major open problem in the field is to characterise the core logic, denoted **IL**(All), that generates the modal logical principles that hold in any reasonable theory.

This paper studies generalised frame conditions for two recently published ([2]) series of principles in **IL**(All). We work with Generalised Veltman semantics (GVS) as introduced by Verbrugge in [8] and defined below, since they allow for a more uniform treatment than regular Veltman semantics (see [6]). For

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example, for various interpretability logics we have completeness with respect to generalised [4] but not with respect to regular Veltman semantics.

Formulas $\mathcal{F}$ of interpretability logic are defined by $\mathcal{F} := \mathsf{Var} \mid \bot \mid \mathcal{F} \to \mathcal{F} \mid \Box\mathcal{F} \mid \mathcal{F} \rhd \mathcal{F}$ where Var is a countable set of propositional variables. Our reading convention stipulates the following binding from strong to weak: $\{\neg, \Box\}$, $\{\wedge, \vee\}$, $\rhd$, $\to$. The $\Box$ modality models formal provability with some base theory T and $A \rhd B$ will stand for "T together with (the interpretation of) A interprets T together (the interpretation of) B". We refer the interested reader to e.g. [3]. We now give the definition of GVS which is similar to regular semantics but now using sets of worlds to model the binary $\rhd$-modality rather than just worlds. In this sense, GVS is reminiscent to neighbourhood semantics.

Definition 1.1 A generalised Veltman frame is a triple $F = \langle W, R, S\rangle$ where the set of worlds W is nonempty, $R \subseteq W^2$ and $S \subseteq W \times W \times (\mathcal{P}(W) \setminus \{\varnothing\})$. We write wRu instead of $\langle w, u\rangle \in R$ and uS_wY instead of $\langle w, u, Y\rangle \in S$. The structure must satisfy the following conditions :

(i) R is transitive and conversely well-founded;
(ii) if uS_wY then wRu and for all $y \in Y$ we have wRy;
(iii) if wRu then $uS_w\{u\}$; and if wRu and uRv then $uS_w\{v\}$;
(iv) if uS_wY and yS_wZ_y for all $y \in Y$, then $uS_w\left(\bigcup_{y\in Y} Z_y\right)$.

Frames extend to models by endowing them with a valuation on the set of propositional variables Var.

Definition 1.2 A generalised Veltman model is a pair $M = \langle F, V\rangle$ with a generalised Veltman frame F and a valuation $V \subseteq W \times \mathsf{Var}$. Given a model M, we define a forcing relation $\Vdash \subseteq W \times \mathsf{Fm}$ for all formulas extending provabilty forcing.

Thus, $\neg(w \Vdash \bot)$; $w \Vdash A \to B$ iff $w \Vdash B$ or $\neg(w \Vdash A)$; $w \Vdash \Box A$ iff $\forall u(wRu \Rightarrow u \Vdash A)$. Finally, we stipulate

$w \Vdash A \rhd B$ iff: if wRu and $u \Vdash A$ then there exists Y such that $Y \Vdash B$ and uS_wY. When we write $Y \Vdash B$ we mean that for all $y \in Y$ we have $y \Vdash B$.

If F is a generalised Veltman frame and A a formula, we write $F \Vdash A$ to denote that for every valuation we have $\langle F, V\rangle \Vdash A$. For a given interpretability principle (a scheme of formulas) X we will denote by $(X)_{\mathsf{gen}}$ a first or higher order formula so that for a frame F we have $F \Vdash \mathsf{X}$ for all instances of X iff F as a first or higher order structure validates $(X)_{\mathsf{gen}}$.

2 Frame conditions

The principle $\mathsf{R} : A \rhd B \to \neg(A \rhd \neg C) \rhd B \wedge \Box C$ was proven to be in **IL**(All) in [1] and [2] extends this to two new series: the so-called narrow and broad

[4] Most notably the logic **IL**R as defined below is complete w.r.t. GVS. However, completenes w.r.t. regular semantics is still open and seems hard. Since **IL**R is the base case of the two series that we consider in this paper, we suspect that GVS is more likely to be useful for said series.

series. Apart from being in **IL**(All), not much more is currently known about the series and this paper constitutes some first progress.

2.1 The narrow series

This series has a more complex frame condition and we only comment on the first new principle in it, R_1: $A \rhd B \to (\neg(A \rhd \neg C) \wedge (D \rhd \Diamond E)) \rhd (B \wedge \Box C \wedge (D \rhd E))$. To state the frame condition we define for E a set that $R^{-1}[E] \coloneqq \{x : \exists y \in E.xRy\}$, and $R^{-1}_x[E] \coloneqq R^{-1}[E] \cap R[x]$.

The $(\mathsf{R}_1)_{\mathrm{gen}}$ condition reads as follows:

$$\forall w, x, u, \mathbb{B}, \mathbb{C}, \mathbb{E}\Big(wRxRuS_w\mathbb{B}, \mathbb{C} \in \mathcal{C}(x,u) \Rightarrow (\exists \mathbb{B}' \subseteq \mathbb{B})\Big(xS_w\mathbb{B}', R[\mathbb{B}'] \subseteq \mathbb{C}, (\forall v \in \mathbb{B}')(\forall c \in \mathbb{C}) (vRcS_xR_x^{-1}[\mathbb{E}] \Rightarrow (\exists \mathbb{E}' \subseteq \mathbb{E})cS_v\mathbb{E}')\Big)\Big).$$

Theorem 2.1 $F \vDash (\mathsf{R}_1)_{gen} \iff F \Vdash R_1.$

Proof.

$\Leftarrow$ We will only include one direction leaving the other as an exercise. Assume for a contradiction that $F \nvDash (\mathsf{R}_1)_{\mathrm{gen}}$. It follows that there exist $w, x, u, \mathbb{B}, \mathbb{C}, \mathbb{E}$ such that $wRxRuS_w\mathbb{B}$, $\mathbb{C} \in \mathcal{C}(x,u)$ and:

$$(\forall \mathbb{B}' \subseteq \mathbb{B})\big(xS_w\mathbb{B}', R[\mathbb{B}'] \subseteq \mathbb{C} \Rightarrow (\exists v \in \mathbb{B}')(\exists c \in \mathbb{C})(\exists Z \subseteq R_x^{-1}[\mathbb{E}].vRcS_xZ, \forall \mathbb{E}' \subseteq \mathbb{E}.c\not S_v\mathbb{E}')\big).$$

Let $\mathcal{V}$ be a family of sets, $\mathcal{V} \coloneqq \{U : U \subseteq \mathbb{B}, xS_wU, R[U] \subseteq \mathbb{C}\}$.

From the condition it follows that for every $U \in \mathcal{V}$ the following is valid:

$$(\exists v_U \in U)(\exists c_U \in \mathbb{C})(\exists Z_U \subseteq R_x^{-1}[\mathbb{E}](v_URc_US_xZ_U, (\forall \mathbb{E}' \subseteq \mathbb{E})c_U\not S_{v_U}\mathbb{E}')).$$

Let us fix such v_U and c_U and Z_U for all $U \in \mathcal{V}$.

Define a valuation such that the following applies: $[\![a]\!] = \{u\}$, $[\![b]\!] = \mathbb{B}$, $[\![c]\!] = \mathbb{C}$, $[\![d]\!] = \{c_U : U \in \mathcal{V}\}$, $[\![e]\!] = \mathbb{E}$. Note that for any formula A we define $[\![A]\!] \coloneqq \{w : w \Vdash A\}$.

By assumption we have $w \Vdash a \rhd b \to (\neg(a \rhd \neg c) \wedge (d \rhd \Diamond e)) \rhd (b \wedge \Box c \wedge (d \rhd e))$.

It is easy to see that $w \Vdash a \rhd b$ and $x \Vdash \neg(a \rhd \neg c)$.

Let us prove $x \Vdash d \rhd \Diamond e$. Let $xRc \Vdash D$. Then $c = c_U$ for some $U \in \mathcal{V}$. From the definition of c_U we have $c_US_xZ_U$, a forcing is defined such that e is true exactly on the set $\mathbb{E}$. Hence $R_x^{-1}[\mathbb{E}] \Vdash \Diamond e$ and since $Z_U \subseteq R_x^{-1}[\mathbb{E}]$ it follows that $x \Vdash d \rhd \Diamond e$.

We can also check that for $U \in \mathcal{V}$ we have $U \Vdash b \wedge \Box c$ and the following condition holds for any set U:

$$(\star)\ xS_wU, U \Vdash b \wedge \Box c \Rightarrow U \in \mathcal{V}.$$

Since $w \Vdash a \rhd b$ and $wRx \Vdash \neg(a \rhd \neg c) \wedge (d \rhd \Diamond e)$ there must exist some set U such that $xS_wU \Vdash b \wedge \Box c \wedge (d \rhd e)$. From $(\star)$ it follows that that $U \in \mathcal{V}$;

hence there exist v_U, c_U, Z_U such that $Z_U \subseteq R_x^{-1}[\mathbb{E}]$ and $v_U R c_U S_x Z_U, (\forall \mathbb{E}' \subseteq \mathbb{E}) c_U \not S_{v_U} \mathbb{E}'$. Since $c_U \Vdash d$ there must exist some Y such that $c_U S_{v_U} Y \Vdash e$, however, by the definition of the valuation it follows that $Y \subseteq \mathbb{E}$ and thus $c_U \not S_{v_U} Y$, which is a contradiction. □

2.2 The broad series

In order to define the R^n principles we first define a series of auxiliary formulas U_k via $\mathsf{U}_0 := \Diamond\neg(D_0 \rhd \neg C)$ and $\mathsf{U}_{r+1} := \Diamond((D_r \rhd D_{r+1}) \wedge \mathsf{U}_r)$. Next, we define

$$\mathsf{R}^0 := A \rhd B \to \neg(A \rhd \neg C) \rhd B \wedge \Box C;$$
$$\mathsf{R}^{n+1} := A \rhd B \to ((D_n \rhd A) \wedge \mathsf{U}_n) \rhd B \wedge \Box C.$$

For $n = 1$ we have $\mathsf{R}^1 = \mathsf{A} \rhd \mathsf{B} \to (\Diamond\neg(\mathsf{D} \rhd \neg\mathsf{C}) \wedge (\mathsf{D} \rhd \mathsf{A})) \rhd (\mathsf{B} \wedge \Box\mathsf{C})$ and the $(\mathsf{R}^1)_{\text{gen}}$ condition reads as follows [5] :

$$\begin{aligned}
&\forall w, x, y, z, \mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{D}.\\
&wRxRyRz,\\
&(\forall u.wRu, u \in \mathbb{A} \Rightarrow \exists V.uS_wV, V \subseteq \mathbb{B}),\\
&(\forall u.xRu, u \in \mathbb{D} \Rightarrow \exists V.uS_xV, V \subseteq \mathbb{A}),\\
&(\forall V.zS_yV \Rightarrow \exists v \in V.v \in \mathbb{C}),\\
&z \in \mathbb{D}\\
\Rightarrow\ &\exists V \subseteq \mathbb{B}(xS_wV, R[V] \subseteq \mathbb{C}).
\end{aligned}$$

We have generalised the previous condition to work for any n. The proof is formalised in Agda and can be found in [4,5]. We proceed by stating the theorem.

Theorem 2.2 $F \vDash (\mathsf{R}^\mathsf{n})_{gen} \iff F \Vdash \mathsf{R}^n$.

3 Agda formalisation

The proofs presented in this paper have been formalised in the Agda ([7]). Agda is a dependently typed language based on an extension of Per Martin-Löf's intuitionistic type theory. Dependent types allow the user to express mathematical properties with types and prove them by providing a term which inhabits such type. Its development mostly takes place at the Chalmers University of Technology.

The presented advances in this paper are part of a broader project ([5]) that aims at establishing a modern and state-of-the-art Agda library for interpretability logics, with a focus on generalised Veltman Semantics. To the best of our knowledge our work is the first attempt at formalising interpretability logics in Agda or any other proof assistant. By the time of this submission the library had around 4000 lines of code and includes, but is not limited to, the following features:

[5] We note that the definition of a scheme being frame valid is second order. As such, a methodological question urges itself (see [5]) in the realm of neighborhood semantics and generalised Veltman semantics: what constitutes a *natural* frame-condition?

- Formalisation of ordinary semantics, generalised semantics and a plethora of useful lemmas to work on such semantics.
- Due to the many possible quasi-transitivity principles available for generalised semantics ([4]) we have defined generalised frames to be parameterized by such condition. All known quasi-transitivity conditions are included in the library and all theorems that do not directly depend on them can be instantiated to work for any quasi-transitivity condition. It also includes a thorough analysis of the interrelations between the alluded conditions.
- We have included proofs for a number of frame conditions. Both for ordinary and generalised semantics. These include M, $\mathsf{P_0}$, R, $\mathsf{M_0}$ for both semantics and $\mathsf{R^n}$, $\mathsf{R_1}$ for generalised semantics.
- The library is not limited to semantics and it includes a definition of the logic **IL**. It also includes an embedded domain specific language to write Hilbert style proofs in a paper-like format. We plan on including derivations of some of the most well known theorems of interpretability logics.

We humbly believe that our library, although under progress is a display of the potential and elegance of Agda. In [5] one can find the full details of the presented theorems in this paper in conjunction with an extensive explanation of the mentioned library. The code is freely available at

`https://gitlab.com/janmasrovira/interpretability-logics`.

References

[1] Goris, E. and J. Joosten, *A new principle in the interpretability logic of all reasonable arithmetical theories*, Logic Journal of the IGPL **19** (2011), pp. 14–17.

[2] Goris, E. and J. J. Joosten, *Two new series of principles in the interpretability logic of all reasonable arithmetical theories*, Journal of Symbolic Logic **85** (2020), pp. 1–25.

[3] Japaridze, G. and D. de Jongh, *The logic of provability*, in: S. Buss, editor, *Handbook of proof theory*, North-Holland Publishing Co., Amsterdam, 1998 pp. 475–546.

[4] Joosten, J. J., J. M. Rovira, L. Mikec and M. Vuković, *An overview of generalised veltman semantics* (2020).
URL `https://arxiv.org/abs/2007.04722`

[5] Mas Rovira, J., "Frame Conditions for Interpretability Logics using Generalised Veltman Semantics and the Agda Proof Assistant," Master's thesis, Master of Pure and Applied Logic, University of Barcelona (forthcoming, 2020).
URL `http://diposit.ub.edu/dspace/handle/2445/133559`

[6] Mikec, L. and M. Vuković, *Interpretability logics and generalised Veltman semantics*, Accepted for publication in The Journal of Symbolic Logic (to appear).
URL `https://arxiv.org/abs/1907.03849(preprintversion)`

[7] Norell, U., "Towards a practical programming language based on dependent type theory," Ph.D. thesis, Department of Computer Science and Engineering, Chalmers University of Technology, SE-412 96 Göteborg, Sweden (2007).

[8] Verbrugge, L., *Verzamelingen-veltman frames en modellen (set veltman frames and models)* (1992), unpublished manuscript.

Expressivity of Linear Temporal Logic under Team Semantics

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Abstract

We develop Kamp-style results for the asynchronous and synchronous variants of linear temporal logic under team semantics. We define a simple translation from the asynchronous semantics to first-order logic under team semantics that uses the flatness of both logics, a property which is lost in some extensions of the logics. We develop the translation further to accommodate for logics that lack flatness, wherein we translate to dependence logic with the classical negation. Finally we formulate the translation from the synchronous semantic to dependence logic with classical negation.

Keywords: Team semantics, linear temporal logic, hyperproperties.

1 Introduction

Linear temporal logic (LTL) is a simple logic for formalising concepts of time. It has become important in theoretical computer science, where Amir Pnueli connected it to system verification in 1977, and within that context the logic has been studied extensively [6]. With regards to expressive power, a classic result by Hans Kamp from 1968 shows that LTL is expressively equivalent to $\mathsf{FO}^2(<)$ [4,7].

LTL has found applications in the field of formal verification, where it is used to check whether a system fulfils its specifications. However, the logic cannot capture all of the interesting specifications a system may have, since it cannot express dependencies between its executions, known as traces. These properties, coined hyperproperties by Clarkson and Schneider in 2010, include properties important for cybersecurity such as noninterference and secure information flow [2]. Due to this background, extensions of LTL have recently been the focus of research.

HyperLTL is one of the most extensively studied of these extensions [1]. Its formulas are interpreted over sets of traces and the syntax extends LTL with quantification on traces. Among the many results for the logic, there are many expressivity results, that relate it to fragments of first, and even second order logic. In particular there is a translation from HyperLTL to $\mathsf{FO}(<, \mathsf{E})$, where

E is an equal level predicate [3]. Here the sets of traces T are coded as $T \times \mathbb{N}$ for the domains of the first-order models.

On the other hand, there are alternative approaches to extending LTL to catch hyperproperties. Team semantics is a framework in which one moves on from considering truth through single assignments to regarding teams of assignments as the linchpin for the satisfaction of a formula. Clearly, this framework, when applied to LTL, provides an approach on the hyperproperties. Krebs et al in 2018 introduced two semantics for LTL under team semantics: the synchronous semantics and the asynchronous variant that differ on the interpretation of the temporal operators [5]. The same paper showed a variety of complexity and expressivity results for the two semantics, as well as that the asynchronous semantic has the flatness property, while the synchronous one does not. This article will follow the semantic definitions of that previous work.

In this article several translations between fragments of TeamLTL and FO under team semantics are introduced. Firstly, we define a translation from the asynchronous semantics to FO^3 under team semantics, which relies on the flatness of both logics. Next we develop this translation further, in order to accommodate for extensions of asynchronous TeamLTL which lack flatness, and we translate them to $\mathsf{FO}^3(=(\dots), \sim)$. We further evolve the previous translation to apply to the synchronous semantics, which in turn we translate to $\mathsf{FO}^4(=(\dots), \sim)$.

Preliminaries

Definition 1.1 [Traces] Let Φ be a set of atomic propositions. A *trace* π over Φ is an infinite sequence $\pi \in (2^\Phi)^\omega$. We denote a trace as $\pi = (\pi(i))_{i=0}^\infty$, and given $j \geq 0$ we denote the suffix of π starting at the jth element $\pi[j, \infty) := (\pi(i))_{i=j}^\infty$.

Definition 1.2 [Linear Temporal Logic] Formulas of LTL are defined by the grammar

$$\varphi := p \mid \neg p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid X\varphi \mid F\varphi \mid G\varphi \mid \varphi U \varphi \mid \varphi R \varphi,$$

where $p \in \Phi$.

Definition 1.3 [Classical Semantics for LTL] Given a trace π, proposition $p \in \Phi$, and LTL formulas φ and ψ, the semantics of linear temporal logic are as follows.

$$\pi \models p \Leftrightarrow p \in \pi(0)$$
$$\pi \models \neg p \Leftrightarrow p \notin \pi(0)$$
$$\pi \models \varphi \wedge \psi \Leftrightarrow \pi \models \varphi \text{ and } \pi \models \psi$$
$$\pi \models \varphi \vee \psi \Leftrightarrow \pi \models \varphi \text{ or } \pi \models \psi$$
$$\pi \models X\varphi \Leftrightarrow \pi[1, \infty) \models \varphi$$
$$\pi \models F\varphi \Leftrightarrow \exists k \geq 0 : \pi[k, \infty) \models \varphi$$
$$\pi \models G\varphi \Leftrightarrow \forall k \geq 0 : \pi[k, \infty) \models \varphi$$
$$\pi \models \varphi U \psi \Leftrightarrow \exists k \geq 0 : \pi[k, \infty) \models \psi \text{ and } \forall k' < k : \pi[k', \infty) \models \varphi$$
$$\pi \models \varphi R \psi \Leftrightarrow \forall k \geq 0 : \pi[k, \infty) \models \psi \text{ or } \exists k' < k : \pi[k', \infty) \models \varphi$$

A team of TeamLTL is a set of traces. We denote $T[i, \infty) := \{t[i, \infty) \mid t \in T\}$. The upcoming definitions are following Krebs et al [5].

Definition 1.4 [Team Semantics for LTL] Suppose T is a team, $p \in \Phi$ is a proposition, and φ and ψ are $\mathsf{TeamLTL}$ formulae. Then the semantics of $\mathsf{TeamLTL}$ are defined by the following.

$T \models p \Leftrightarrow p \in \pi(0)$ for all $\pi \in T$
$T \models \neg p \Leftrightarrow p \notin \pi(0)$ for all $\pi \in T$
$T \models \varphi \wedge \psi \Leftrightarrow T \models \varphi$ and $T \models \psi$
$T \models \varphi \vee \psi \Leftrightarrow \exists T_1, T_2 \subseteq T :$
$T_1 \cup T_2 = T$ and
$T_1 \models \varphi$ and $T_2 \models \psi$
$T \models X\varphi \Leftrightarrow T[1, \infty) \models \varphi$
$T \models F^s\varphi \Leftrightarrow \exists k \geq 0 :$
$T[k, \infty) \models \varphi$
$T \models F^a\varphi \Leftrightarrow \forall \pi \in T\ \exists k_\pi \geq 0 :$
$\{\pi[k_\pi, \infty) \mid \pi \in T\} \models \varphi$
$T \models G^s\varphi \Leftrightarrow \forall k \geq 0 : T[k, \infty) \models \varphi$
$T \models G^a\varphi \Leftrightarrow \forall \pi \in T$ and
$\forall k_\pi \geq 0\ \{\pi[k_\pi, \infty) \mid \pi \in T\} \models \varphi$
$T \models \varphi U^s \psi \Leftrightarrow \exists k \geq 0 : T[k, \infty) \models \psi$
and $\forall k' < k : T[k', \infty) \models \varphi$
$T \models \varphi U^a \psi \Leftrightarrow \forall \pi \in T\ \exists k_\pi \geq 0 :$
$\{\pi[k_\pi, \infty) \mid \pi \in T\} \models \psi$ and
$\forall k'_\pi < k_\pi : \{\pi[k'_\pi, \infty) \mid \pi \in T\} \models \varphi$
$T \models \varphi R^s \psi \Leftrightarrow \forall k \geq 0 :$
$T[k, \infty) \models \psi$ or $\exists k' < k :$
$T[k', \infty) \models \varphi$
$T \models \varphi R^a \psi \Leftrightarrow \forall \pi \in T\ \forall k_\pi \geq 0 :$
$\{\pi[k_\pi, \infty) \mid \pi \in T\} \models \psi$ or
$\exists k'_\pi < k_\pi : \{\pi[k'_\pi, \infty) \mid \pi \in T\} \models \varphi$

We denote the asynchronous and the synchronous fragments by $\mathsf{TeamLTL}^{\mathsf{a}}$ and $\mathsf{TeamLTL}^{\mathsf{s}}$, respectively.

Definition 1.5 [FO under team semantics and $\mathsf{FO}(=(\dots), \sim)$] Formulae of FO are defined by the grammar

$$\varphi := x = y \mid R(x_1, \dots, x_n) \mid \neg x = y \mid \neg R(x_1, \dots, x_k) \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \exists x\varphi \mid \forall x\varphi,$$

where x, y and $x_1, \dots, x_n$ are variables, and R is a relation symbol of arity n. Formulas of $\mathsf{FO}(=(\dots), \sim)$ extend the grammar by the dependence atom $=(x_1, \dots, x_n, y)$ and the Boolean negation $\sim \varphi$.

In the following we use the notation $T[F/x] = \{t[F(t)/x] \mid t \in T\}$, where T is a team, $F : T \to \mathcal{P}(M) \setminus \emptyset$ is a supplementation function and x is a variable. Similarly we notate duplication through $T[M/x] = \{t[m/x] \mid$ for all $m \in M, t \in T\}$.

Definition 1.6 [Team Semantics for FO] Suppose $\mathcal{M}$ is a first-order model with domain M, and let S be a team of $\mathcal{M}$. Suppose $n \geq 1$, and φ and ψ are FO formulae. Then the team semantics of FO are defined by the following.

$\mathcal{M} \models_S x = y \Leftrightarrow \forall s \in S, s(x) = s(y)$
$\mathcal{M} \models_S R(x_1, \dots, x_n) \Leftrightarrow \forall s \in S,$
$(s(x_1), \dots, s(x_n)) \in R^{\mathcal{M}}$
$\mathcal{M} \models_S \neg x = y \Leftrightarrow \forall s \in S, s(x) \neq s(y)$
$\mathcal{M} \models_S \neg R(x_1, \dots, x_n) \Leftrightarrow \forall s \in S,$
$(s(x_1), \dots, s(x_n)) \notin R^{\mathcal{M}}$
$\mathcal{M} \models_S \varphi \wedge \psi \Leftrightarrow \mathcal{M} \models_S \varphi$ and
$\mathcal{M} \models_S \psi$
$\mathcal{M} \models_S \varphi \vee \psi \Leftrightarrow \exists S_1, S_2 \subseteq S$
such that $S_1 \cup S_2 = S$ and
$\mathcal{M} \models_{S_1} \varphi$ and $\mathcal{M} \models_{S_2} \psi$
$\mathcal{M} \models_S \exists x\varphi \Leftrightarrow \exists F : S \to \mathcal{P}(M) \setminus \emptyset$
such that $\mathcal{M} \models_{S[F/x]} \varphi$
$\mathcal{M} \models_S \forall x\varphi \Leftrightarrow \mathcal{M} \models_{S[M/x]} \varphi$

The semantics of $\mathsf{FO}(=(\dots), \sim)$ extend the previous with the following.

$\mathcal{M} \models_S =(x_1, \dots, x_n, y) \Leftrightarrow \forall s_1, s_2 \in S,$ if $s_1(x_i) = s_2(x_i)$ for all $i \in \{1, \dots, n\}$, then $s_1(y) = s_2(y)$
$\mathcal{M} \models_S \sim \varphi \Leftrightarrow \mathcal{M} \not\models_S \varphi$

We say that FO under team semantics has the *flatness property*, since for all formulae φ of FO, models $\mathcal{M}$ and teams T it holds that $\mathcal{M} \models_T \varphi$ if and only if $\mathcal{M} \models_{\{t\}} \varphi$ for all $t \in T$. Similarly for all formulae φ of $\mathsf{TeamLTL}$ and teams T, $T \models \varphi$ if and only if $\{\pi\} \models \varphi$ for all $\pi \in T$.

A Translation of Asynchronous LTL to FO

Suppose $T = \{\pi_j \mid j \in J\}$ is a team of traces. Define $\mathcal{M}_T$ to be the following structure of vocabulary $\{\leq\} \cup \{P_i \mid p_i \in \Phi\}$ where

$$Dom(\mathcal{M}_T) = T \times \mathbb{N}$$
$$\leq^{\mathcal{M}_T} = \{((\pi_i, n), (\pi_j, m)) \mid i = j \text{ and } n \leq m\}$$
$$P_i^{\mathcal{M}_T} = \{(\pi_k, j) \mid p_i \in \pi_k(j)\}.$$

In addition we define a team $S_T = \{s_i \mid s_i(x) = (\pi_i, 0), \text{ for all } \pi_i \in T\}$. We notate $\varphi \hookrightarrow \psi := \neg\varphi \vee (\varphi \wedge \psi)$.

Next we define inductively the translations ST_w, where $w \in \{x, y, z\}$, from $\mathsf{TeamLTL}^{\mathsf{a}}$ to FO^3 under team semantics as follows:

$ST_x(p_i) = P_i(x)$
$ST_x(\neg p_i) = \neg P_i(x)$
$ST_x(\varphi \wedge \psi) = ST_x(\varphi) \wedge ST_x(\psi)$
$ST_x(\varphi \vee \psi) = ST_x(\varphi) \vee ST_x(\psi)$
$ST_x(X\varphi) = \exists y(x < y \wedge ST_y(\varphi) \wedge \forall z \neg(x < z \wedge z < y))$
$ST_x(G^a\varphi) = \forall y(x \leq y \hookrightarrow ST_y(\varphi))$
$ST_x(F^a\varphi) = \exists y(x \leq y \wedge ST_y(\varphi))$
$ST_x(\varphi U^a \psi) = \exists y(x \leq y \wedge ST_y(\psi) \wedge \forall z((x \leq z \wedge z < y) \hookrightarrow ST_z(\varphi)))$
$ST_x(\varphi R^a \psi) = \forall y(x \leq y \hookrightarrow (ST_y(\psi) \vee \exists z(x \leq z \wedge z < y \wedge ST_z(\varphi)))).$

Proposition 1.7 *For all* $\mathsf{TeamLTL}^{\mathsf{a}}$ *formulae* φ, $T \models \varphi \Leftrightarrow \mathcal{M}_T \models_{S_T} ST_x(\varphi)$.

This proposition follows from the fact that both logics are flat, and in fact, by the same argument, any translation from LTL to FO is also a translation for the asynchronous semantic.

Translations in the Absence of Flatness

The previous translation makes use of the fact that both $\mathsf{TeamLTL}^{\mathsf{a}}$ and FO have the flatness property. However, flatness does not hold for $\mathsf{TeamLTL}^{\mathsf{s}}$ or extensions of $\mathsf{TeamLTL}^{\mathsf{a}}$. Thus the translation needs to be modified to accommodate for these cases. To that end let $\mathcal{M}_T$ and S_T be as previously. We define a translation of $\mathsf{TeamLTL}^{\mathsf{a}}$ formulas to $\mathsf{FO}^3(=(\dots), \sim)$ as follows:

The translation is analogous to the previous translation for the atomic propositions, $\wedge$, $\vee$, and X.

$$ST^*_x(F^a\varphi) = \exists y(x \leq y \wedge {=}(x,y) \wedge ST^*_y(\varphi))$$
$$ST^*_x(G^a\varphi) = \sim \exists y(x \leq y \wedge {=}(x,y) \wedge \sim ST^*_y(\varphi))$$
$$ST^*_x(\varphi U^a\psi) = \exists y(x \leq y \wedge {=}(x,y) \wedge ST^*_y(\psi) \wedge \sim \exists z(x \leq z \wedge z \leq y \wedge {=}(x,z) \wedge \sim ST^*_z(\varphi)))$$
$$ST^*_x(\varphi R^a\psi) = \sim \exists y(x \leq y \wedge {=}(x,y) \wedge \sim ST^*_y(\psi) \wedge \exists z(x \leq z \wedge z < y \wedge {=}(x,z) \wedge \sim ST^*_z(\varphi))).$$

Theorem 1.8 *For all* $\mathsf{TeamLTL^a}$ *formulae* φ *there exists a* $\mathsf{FO}^3({=}(\ldots), \sim)$ *formula* $ST^*_x(\varphi)$*, such that* $T \models \varphi \Leftrightarrow \mathcal{M}_T \models_{S_T} ST^*_x(\varphi)$.

This result can now easily be expanded to extensions of $\mathsf{TeamLTL^a}$ which do not have the flatness property, by providing a translation for the extending atoms or operators. For instance, the dependence atom satisfies the equivalence ${=}(p,q) \equiv (p \wedge (q \varovee \neg q) \vee (\neg p \wedge (q \varovee \neg q)$, which uses the Boolean disjunction $\varovee$ that can be expressed in $\mathsf{FO}({=}(\ldots), \sim)$. Thus by using this equivalence we can translate any formula of $\mathsf{TeamLTL^a}({=}(\ldots))$ to $\mathsf{FO}^3({=}(\ldots), \sim)$ using the previous translation.

Corollary 1.9 *For all* $\mathsf{TeamLTL^a}({=}(\ldots))$ *formulae* φ, $T \models \varphi \Leftrightarrow \mathcal{M}_T \models_{S_T} ST^*_x(\varphi)$.

Translation for Synchronous TeamLTL

The synchronous team semantics for LTL does not have the flatness property [5]. Armed with the previous translation, we need to capture the equal level teams on the first-order side. This can be done as for HyperLTL, by introducing an equal level predicate E [3].

Let $\mathcal{M}_T$ and S_T be as above, with the addition of the equal level predicate E together with its negation, both defined in the usual way by $E^{\mathcal{M}_T} = \{((\pi_i,k),(\pi_j,k)) \mid i,j \in J \text{ and } k \in \mathbb{N}\}$. Next we define a translation from $\mathsf{TeamLTL^s}$ to $\mathsf{FO}^4({=}(\ldots), \sim)$ as follows: The translation is analogous to the previous translations for the atomic propositions, $\wedge$, $\vee$, and X.

$$ST^*_x(F^s\varphi) = \exists y({=}(y) \wedge \exists z(E(y,z) \wedge x \leq z \wedge ST^*_z(\varphi)))$$

The remaining operators are translated in a similar way to the future operator, while using the pattern established in the previous translations.

Theorem 1.10 *For all* $\mathsf{TeamLTL^s}$ *formulae* φ *there exists a* $\mathsf{FO}^4({=}(\ldots), \sim)$ *formula* $ST^*_x(\varphi)$*, such that* $T \models \varphi \Leftrightarrow \mathcal{M}_T \models_{S_T} ST^*_x(\varphi)$.

The translations presented in this article fill in parts of the web of expression of TeamLTL. In future research the translations can be used to further study the expressivity and complexity of TeamLTL and its extensions, for instance the precise fragments of the first-order logics that correspond to the temporal team logics remains to be determined. The connections between the fragments of FO under team semantics and HyperFO implied by the similarity of the two constructions also provide questions for further research.

References

[1] Clarkson, M. R., B. Finkbeiner, M. Koleini, K. K. Micinski, M. N. Rabe and C. Sánchez, *Temporal logics for hyperproperties*, in: M. Abadi and S. Kremer, editors, *Principles of Security and Trust - Third International Conference, POST, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS Proceedings*, Lecture Notes in Computer Science **8414** (2014), pp. 265–284.

[2] Clarkson, M. R. and F. B. Schneider, *Hyperproperties*, Journal of Computer Security **18** (2010), pp. 1157–1210.

[3] Finkbeiner, B. and M. Zimmermann, *The first-order logic of hyperproperties*, in: H. Vollmer and B. Vallée, editors, *34th Symposium on Theoretical Aspects of Computer Science, STACS*, LIPIcs **66** (2017), pp. 30:1–30:14.

[4] Kamp, H. W., "Tense logic and the theory of linear order," Ph.D. thesis, University of California, USA (1968).

[5] Krebs, A., A. Meier, J. Virtema and M. Zimmermann, *Team semantics for the specification and verification of hyperproperties*, in: I. Potapov, P. G. Spirakis and J. Worrell, editors, *43rd International Symposium on Mathematical Foundations of Computer Science, MFCS*, LIPIcs **117** (2018), pp. 10:1–10:16.

[6] Pnueli, A., *The temporal logic of programs*, in: *18th Annual Symposium on Foundations of Computer Science* (1977), pp. 46–57.

[7] Rabinovich, A., *A proof of Kamp's theorem*, Logical Methods in Computer Science **10** (2014).

Subformula Property and Craig Interpolation Theorem of Sequent Calculi for Tense Logics

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Abstract

This paper establishes the subformula property and the Craig interpolation theorem for sequent calculi of the tense expansions of modal logics **K**, **KT**, **KD**, **K4**, **K4D**, and **S4**. Our sequent calculi are based on the ordinary notion of (non-labelled) sequent. We prove the subformula property of all the calculi by Takano's semantic argument and apply Maehara method to get the Craig interpolation theorem.

Keywords: Analytic Cut, Craig Interpolation, Subformula Property, Sequent Calculus, Tense Logic

1 Introduction and Motivation

If we focus on the modal cube, i.e., the fifteen distinct modal logics generated from modal axioms **D**, **T**, **B**, **4** and **5**, all the modal logics have the corresponding sequent calculi and the calculi enjoy the subformula property [5,7], though we need to extend the notion of subformula for modal logics **K5** and **K5D** [6]. While it is well-known that some sequent calculi for the fifteen modal logics (say, for **S5**) do not enjoy the cut-elimination theorem, Takano proposed that the subformula property can be regarded as a substitute of cut-elimination for modal logics. A key ingredient of this claim is that all applications of the cut rule can be replaced with analytic applications of the cut rule, i.e., applications where the cut formula is a subformula in a formula of the conclusion

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of the cut rule. Recently, Kowalski and Ono [1] extended this perspective to bi-intuitionistic logic to get the Craig interpolation theorem.

This paper extends Takano's claim also to tense expansions of modal logics. In particular, we provide sequent calculi of the tense expansions of modal logics **K**, **KT**, **KD**, **K4**, **K4D** and **S4**, and then establish (semantically) that all the calculi enjoy the subformula property (with the help of an analytic cut rule). As a corollary of the subformula property, we establish the Craig interpolation theorem by Maehara method via the ordinary notion of sequent. This contrasts with the recent result [2, Corollary 16] of the Craig interpolation theorem for tense logics in terms of the notion of nested (or labelled) sequent.

2 Sequent Calculi for Tense Logics

Let Prop be a countable set of propositional variables. Our syntax $\mathcal{L}$ for tense logic consists of Prop and all logical connectives of classical logic, i.e., a constant symbol $\bot$, negation $\neg$, disjunction $\vee$, conjunction $\wedge$, implication $\rightarrow$, as well as two modal operators $\{\blacklozenge, \Box\}$, where $\Box$ is the future necessity operator and $\blacklozenge$ is the past possibility operator. The set of all *formulas* in $\mathcal{L}$ is defined in a standard way. Given any formula φ, we define $\mathsf{Sub}(\varphi)$ as the set of all subformulas of φ. Moreover, for any set (or multiset) Γ of formulas, we define $\mathsf{Sub}(\Gamma) = \bigcup_{\varphi \in \Gamma} \mathsf{Sub}(\varphi)$. We say that a set (or multiset) Γ is *subformula closed* if $\mathsf{Sub}(\varphi) \subseteq \Gamma$ for all formulas $\varphi \in \Gamma$.

Given a *Kripke frame* (W, R) (where W is a non-empty set and R is a binary relation on W), we follow the standard definitions for frame properties of R such as reflexivity, transitivity, and seriality. A *Kripke model* $M = (W, R, V)$ consists of a Kripke frame (W, R) and a *valuation* $V : \mathsf{Prop} \rightarrow \wp(W)$. Given a model $M = (W, R, V)$ and a state $w \in W$, a *satisfaction relation* $M, w \models \varphi$ (read "φ is true at w of M") is defined inductively as usual, in particular,

$$\begin{array}{ll} M, w \models \Box\varphi \text{ iff} & \text{for every } v,\ wRv \text{ implies } M, v \models \varphi, \\ M, w \models \blacklozenge\varphi \text{ iff} & \text{for some } v,\ vRw \text{ and } M, v \models \varphi. \end{array}$$

We say that a formula φ is *valid* on a class $\mathbb{M}$ of Kripke models if, for every Kripke model M in $\mathbb{M}$, $M, w \models \varphi$ holds for all states w in M.

Table 1
Sequent Rules for Tense Operators

$$\frac{\blacklozenge\Theta, \Pi \Rightarrow \varphi}{\Theta, \Box\Pi \Rightarrow \Box\varphi}\ (\Box) \qquad \frac{\varphi \Rightarrow \Sigma, \Box\Theta}{\blacklozenge\varphi \Rightarrow \blacklozenge\Sigma, \Theta}\ (\blacklozenge) \qquad \frac{\blacklozenge\Theta, \Pi \Rightarrow}{\Theta, \Box\Pi \Rightarrow}\ (\Box_{\mathbf{D}})$$

$$\frac{\varphi, \Gamma \Rightarrow \Delta}{\Box\varphi, \Gamma \Rightarrow \Delta}\ (\Box \Rightarrow) \qquad \frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \blacklozenge\varphi}\ (\Rightarrow \blacklozenge)$$

$$\frac{\blacklozenge\Omega, \blacklozenge\Theta, \Box\Pi, \Pi \Rightarrow \varphi}{\blacklozenge\Omega, \Theta, \Box\Pi \Rightarrow \Box\varphi}\ (\Box_{\mathbf{4}}) \qquad \frac{\varphi \Rightarrow \Sigma, \blacklozenge\Sigma, \Box\Theta, \Box\Omega}{\blacklozenge\varphi \Rightarrow \blacklozenge\Sigma, \Theta, \Box\Omega}\ (\blacklozenge_{\mathbf{4}}) \qquad \frac{\blacklozenge\Omega, \blacklozenge\Theta, \Box\Pi, \Pi \Rightarrow}{\blacklozenge\Omega, \Theta, \Box\Pi \Rightarrow}\ (\Box_{\mathbf{4D}})$$

$$\frac{\blacklozenge\Theta, \Box\Pi \Rightarrow \varphi}{\blacklozenge\Theta, \Box\Pi \Rightarrow \Box\varphi}\ (\Rightarrow \Box) \qquad \frac{\varphi \Rightarrow \blacklozenge\Sigma, \Box\Theta}{\blacklozenge\varphi \Rightarrow \blacklozenge\Sigma, \Box\Theta}\ (\blacklozenge \Rightarrow)$$

In what follows, we use Γ, Δ, etc. to denote finite multisets of formulas. A *sequent* is a pair of finite multisets and it is denoted by $\Gamma \Rightarrow \Delta$, where Γ and Δ are called an *antecedent* and a *succedent*, respectively. We read $\Gamma \Rightarrow \Delta$ as "if all formulas in Γ hold, then some formulas in Δ holds". Let **LK** be a set of initial sequents ($\varphi \Rightarrow \varphi$ and $\bot \Rightarrow$), structural rules (right and left rules for contraction and weakening), propositional rules (right and left rules for $\neg$, $\wedge$, $\vee$, and $\rightarrow$) and the rule of *cut*:

$$\frac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma} \ (Cut).$$

Definition 2.1 Let $\Lambda \in \{\mathbf{K}, \mathbf{KD}, \mathbf{KT}, \mathbf{K4}, \mathbf{K4D}, \mathbf{S4}\}$. A sequent calculus $\mathsf{G}(\Lambda_{\mathbf{t}})$ is defined in terms of rules given in Table 1 as follows:

- $\mathsf{G}(\mathbf{K_t})$ consists of **LK**, ($\Box$) and ($\blacklozenge$). (cf. [3])
- $\mathsf{G}(\mathbf{KD_t})$ is the expansion of $\mathsf{G}(\mathbf{K_t})$ with ($\Box_{\mathbf{D}}$).
- $\mathsf{G}(\mathbf{KT_t})$ is the expansion of $\mathsf{G}(\mathbf{K_t})$ with ($\Box \Rightarrow$) and ($\Rightarrow \blacklozenge$).
- $\mathsf{G}(\mathbf{K4_t})$ consists of **LK**, ($\Box_{\mathbf{4}}$) and ($\blacklozenge_{\mathbf{4}}$). (cf. [3])
- $\mathsf{G}(\mathbf{K4D_t})$ is the expansion of $\mathsf{G}(\mathbf{K4_t})$ with ($\Box_{\mathbf{4D}}$).
- $\mathsf{G}(\mathbf{S4_t})$ consists of **LK**, ($\Rightarrow \Box$), ($\Box \Rightarrow$), ($\Rightarrow \blacklozenge$) and ($\blacklozenge \Rightarrow$).

For each sequent calculi in Definition 2.1, we define the notions of *proof* and *provable* sequent as usual. The reader may wonder if (Cut) is admissible in all the calculi in Definition 2.1. However, this is not the case. Let us focus on $\mathsf{G}(\mathbf{K_t})$ here. A sequent $p, \blacklozenge\Box\neg p \Rightarrow$ is provable in the calculus with the help of (Cut), but the application of (Cut) is indispensable for the purpose:

$$\frac{\dfrac{\Box\neg p \Rightarrow \Box\neg p}{\blacklozenge\Box\neg p \Rightarrow \neg p}\ (\blacklozenge) \quad \dfrac{p \Rightarrow p}{\neg p, p \Rightarrow}\ (\neg\Rightarrow)}{p, \blacklozenge\Box\neg p \Rightarrow}\ (Cut),$$

where $(\neg\Rightarrow)$ is the left rule for negation (this kind of phenomena is well-known for a sequent calculus of modal logic **S5**, see, e.g., [4, p.222]). It is remarked in the above proof that the cut formula $\neg p$ is a subformula of the conclusion of (Cut) and moreover $\Box\neg p$ is also a subformula of the conclusion of the rule ($\blacklozenge$). Therefore, all the applications of the inference rules in the proof above are *analytic*, i.e., they satisfy the subformula property. This motivates us to define the following analytic variants to the calculi of Definition 2.1.

Definition 2.2 When $\Lambda \in \{\mathbf{K}, \mathbf{KD}, \mathbf{KT}, \mathbf{K4}, \mathbf{K4D}, \mathbf{S4}\}$, $\mathsf{G}^a(\Lambda_{\mathbf{t}})$ is the same system as $\mathsf{G}(\Lambda_{\mathbf{t}})$ except we replace (Cut) by the following analytic cut rule:

$$\frac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma} \ (Cut)^a \text{ where } \varphi \in \mathsf{Sub}(\Gamma, \Pi, \Delta, \Sigma),$$

and we also replace tense logical rules with analytic variants (with superscript "a") requiring the following side conditions:

- $(\Box)^a$: $\blacklozenge\Theta \subseteq \mathsf{Sub}(\Pi, \varphi)$.
- $(\blacklozenge)^a$: $\Box\Theta \subseteq \mathsf{Sub}(\varphi, \Sigma)$.
- $(\Box_{\mathbf{D}})^a$: $\blacklozenge\Theta \subseteq \mathsf{Sub}(\Pi)$.
- $(\Box_{\mathbf{4}})^a$: $\blacklozenge\Theta \subseteq \mathsf{Sub}(\blacklozenge\Omega, \Pi, \varphi)$.
- $(\blacklozenge_{\mathbf{4}})^a$: $\Box\Theta \subseteq \mathsf{Sub}(\varphi, \Sigma, \Box\Omega)$.
- $(\Box_{\mathbf{4D}})^a$: $\blacklozenge\Theta \subseteq \mathsf{Sub}(\blacklozenge\Omega, \Pi)$.

A sequent $\Gamma \Rightarrow \Delta$ is *valid* in a Kripke model M if $\bigwedge\Gamma \to \bigvee\Delta$ is valid in M, where $\bigwedge\Gamma$ and $\bigvee\Delta$ are the conjunction and disjunction of all formulas in Γ and Δ, respectively. Given any $\Lambda \in \{\mathbf{K}, \mathbf{KD}, \mathbf{KT}, \mathbf{K4}, \mathbf{K4D}, \mathbf{S4}\}$, the class $\mathbb{M}^{\text{fin}}_{\Lambda}$ is the class of all *finite* Kripke models whose binary relation R satisfies the corresponding frame properties to Λ.

Proposition 2.3 (Soundness) *Let $\Lambda \in \{\mathbf{K}, \mathbf{KD}, \mathbf{KT}, \mathbf{K4}, \mathbf{K4D}, \mathbf{S4}\}$. If a sequent $\Gamma \Rightarrow \Delta$ is provable in $\mathsf{G}(\Lambda_{\mathbf{t}})$, then it is valid on all models in $\mathbb{M}^{\text{fin}}_{\Lambda}$.*

3 Subformula Property

Let $\Lambda \in \{\mathbf{K}, \mathbf{KD}, \mathbf{KT}, \mathbf{K4}, \mathbf{K4D}, \mathbf{S4}\}$. This section establishes that a fully analytic calculus $\mathsf{G}^a(\Lambda_{\mathbf{t}})$ enjoys the subformula property by showing that $\mathsf{G}^a(\Lambda_{\mathbf{t}})$ is semantically complete for the intended class of finite models. In what follows in this section, we fix Ξ as a subformula closed finite set.

Definition 3.1 We say that a pair (Π, Σ) of finite sets of formulas is a Ξ-*partial valuation* in $\mathsf{G}^a(\Lambda_{\mathbf{t}})$ if the following three conditions are satisfied: (i) $\Pi \Rightarrow \Sigma$ is unprovable in $\mathsf{G}^a(\Lambda_{\mathbf{t}})$, (ii) $\Pi \cup \Sigma = \mathsf{Sub}(\Pi, \Sigma)$, (iii) $\mathsf{Sub}(\Pi, \Sigma) \subseteq \Xi$.

We use the analytic cut rule $(Cut)^a$ to get the following lemma.

Lemma 3.2 *Let $\Gamma \Rightarrow \Delta$ be unprovable in $\mathsf{G}^a(\Lambda_{\mathbf{t}})$. For any subformula closed set Ξ such that $\mathsf{Sub}(\Gamma, \Delta) \subseteq \Xi$, there exists a Ξ-partial valuation (Γ^+, Δ^+) in $\mathsf{G}^a(\Lambda_{\mathbf{t}})$ such that $\Gamma \subseteq \Gamma^+ \subseteq \mathsf{Sub}(\Gamma, \Delta)$ and $\Delta \subseteq \Delta^+ \subseteq \mathsf{Sub}(\Gamma, \Delta)$.*

Definition 3.3 Define $M^{\Xi}_{\Lambda} = (W^{\Xi}, R^{\Xi}_{\Lambda}, V^{\Xi})$ by:

- $W^{\Xi} := \{ (\Pi, \Sigma) \mid (\Pi, \Sigma)$ is a Ξ-partial valuation in $\mathsf{G}^a(\Lambda_{\mathbf{t}})\ \}$.
- R^{Ξ}_{Λ} is defined depending on our choice of Λ as follows:
  - · $(\Gamma, \Delta) R^{\Xi}_{\Lambda} (\Pi, \Sigma)$ iff $\{ \psi \mid \Box\psi \in \Gamma \} \subseteq \Pi$ and $\{ \psi \mid \blacklozenge\psi \in \Sigma \} \subseteq \Delta$, if $\Lambda \in \{\mathbf{K}, \mathbf{KT}, \mathbf{KD}\}$;
  - · $(\Gamma, \Delta) R^{\Xi}_{\Lambda} (\Pi, \Sigma)$ iff $\{ \psi, \Box\psi \mid \Box\psi \in \Gamma \} \subseteq \Pi$ and $\{ \psi, \blacklozenge\psi \mid \blacklozenge\psi \in \Sigma \} \subseteq \Delta$, if $\Lambda \in \{\mathbf{K4}, \mathbf{K4D}\}$;
  - · $(\Gamma, \Delta) R^{\Xi}_{\mathbf{S4}} (\Pi, \Sigma)$ iff $\{ \Box\psi \mid \Box\psi \in \Gamma \} \subseteq \Pi$ and $\{ \blacklozenge\psi \mid \blacklozenge\psi \in \Sigma \} \subseteq \Delta$.
- $(\Gamma, \Delta) \in V^{\Xi}(p)$ iff $p \in \Gamma$.

Lemma 3.4 *For every $(\Gamma, \Delta) \in W^{\Xi}$ and every $\chi \in \Gamma \cup \Delta$, the following hold:* (i) *$\chi \in \Gamma$ implies $M^{\Xi}_{\Lambda}, (\Gamma, \Delta) \models \chi$, and* (ii) *$\chi \in \Delta$ implies $M^{\Xi}_{\Lambda}, (\Gamma, \Delta) \not\models \chi$.*

Lemma 3.5 *For every choice of Λ, the Kripke model M^{Ξ}_{Λ} belongs to $\mathbb{M}_{\Lambda}$.*

Theorem 3.6 *Let $\Lambda \in \{\mathbf{K}, \mathbf{KD}, \mathbf{KT}, \mathbf{K4}, \mathbf{K4D}, \mathbf{S4}\}$. If a sequent $\Gamma \Rightarrow \Delta$ is valid in the class $\mathbb{M}^{\text{fin}}_{\Lambda}$, then it is provable in $\mathsf{G}^a(\Lambda_{\mathbf{t}})$.*

Definition 3.7 For any $\Lambda \in \{\mathbf{K}, \mathbf{KD}, \mathbf{KT}, \mathbf{K4}, \mathbf{K4D}, \mathbf{S4}\}$, we define $\mathsf{G}^*(\Lambda_{\mathbf{t}})$ as the same system as $\mathsf{G}(\Lambda_{\mathbf{t}})$ except that the cut rule (Cut) is replaced by the analytic variant $(Cut)^a$ (recall Definition 2.2).

We establish the Craig interpolation theorem for $\mathsf{G}^*(\Lambda_{\mathbf{t}})$ in the next section, though it is noted that $\mathsf{G}^a(\mathbf{S4_t})$ and $\mathsf{G}^*(\mathbf{S4_t})$ are exactly the same calculi.

Corollary 3.8 *For any sequent* $\Gamma \Rightarrow \Delta$*, the following are all equivalent:* (1) $\Gamma \Rightarrow \Delta$ *is valid in the class* $\mathbb{M}_{\Lambda}^{\mathrm{fin}}$*,* (2) $\Gamma \Rightarrow \Delta$ *is provable in* $\mathsf{G}^a(\Lambda_{\mathbf{t}})$*,* (3) $\Gamma \Rightarrow \Delta$ *is provable in* $\mathsf{G}^*(\Lambda_{\mathbf{t}})$*,* (4) $\Gamma \Rightarrow \Delta$ *is provable in* $\mathsf{G}(\Lambda_{\mathbf{t}})$*.*

Proof. The direction from (1) to (2) is due to Theorem 3.6 and the direction from (4) to (1) is due to Proposition 2.3. The remaining directions (from (2) to (3) and from (3) to (4)) are immediate by definition. □

4 Craig Interpolation Theorem for Tense Logics

This section establishes the Craig interpolation theorem of $\mathsf{G}(\Lambda_{\mathbf{t}})$ for all choices of $\Lambda \in \{\mathbf{K}, \mathbf{KD}, \mathbf{KT}, \mathbf{K4}, \mathbf{K4D}, \mathbf{S4}\}$ by Maehara's method. For this purpose, it suffices to make use of $\mathsf{G}^*(\Lambda_{\mathbf{t}})$ from Definition 3.7, instead of the fully analytic calculus $\mathsf{G}^a(\Lambda_{\mathbf{t}})$. Given any finite multiset Δ, we use $\mathsf{Prop}(\Delta)$ to mean the set of all propositional variables in a formula of Δ. A pair $\langle(\Gamma_1, \Delta_1), (\Gamma_2, \Delta_2)\rangle$ is said to be a *partition* of a sequent $\Gamma \Rightarrow \Delta$ if $\Gamma = \Gamma_1, \Gamma_2$ and $\Delta = \Delta_1, \Delta_2$ and we write it as $(\Gamma_1 : \Delta_1), (\Gamma_2 : \Delta_2)$.

Lemma 4.1 *If a sequent* $\Gamma \Rightarrow \Delta$ *is provable in* $\mathsf{G}^*(\Lambda_{\mathbf{t}})$*, then every partition* $(\Gamma_1 : \Delta_1), (\Gamma_2 : \Delta_2)$ *of* $\Gamma \Rightarrow \Delta$ *has an interpolant, i.e., there exists a formula* θ *such that sequents* $\Gamma_1 \Rightarrow \Delta_1, \theta$ *and* $\theta, \Gamma_2 \Rightarrow \Delta_2$ *are provable in* $\mathsf{G}^*(\Lambda_{\mathbf{t}})$*, and* $\mathsf{Prop}(\theta) \subseteq \mathsf{Prop}(\Gamma_1, \Delta_1) \cap \mathsf{Prop}(\Gamma_2, \Delta_2)$*.*

Proof. By induction on a proof of $\Gamma \Rightarrow \Delta$ in $\mathsf{G}^*(\Lambda_{\mathbf{t}})$. When the last applied rule is $(Cut)^a$, we can apply the same argument as given in [4, pp.245-246]. □

By Corollary 3.8 and Lemma 4.1, the following holds (cf. [2, Corollary 16]).

Theorem 4.2 *Let* $\Lambda \in \{\mathbf{K}, \mathbf{KD}, \mathbf{KT}, \mathbf{K4}, \mathbf{K4D}, \mathbf{S4}\}$*. If a sequent* $\Rightarrow \varphi \to \psi$ *is provable in* $\mathsf{G}(\Lambda_{\mathbf{t}})$ *then there exists a formula* θ *such that both sequents* $\Rightarrow \varphi \to \theta$ *and* $\Rightarrow \theta \to \psi$ *are provable in* $\mathsf{G}(\Lambda_{\mathbf{t}})$*, and* $\mathsf{Prop}(\theta) \subseteq \mathsf{Prop}(\varphi) \cap \mathsf{Prop}(\psi)$*.*

References

[1] Kowalski, T. and H. Ono, *Analytic cut and interpolation for bi-intuitionistic logic*, The Review of Symbolic Logic **10(2)** (2017), pp. 259–283.

[2] Lyon, T., A. Tiu, R. Goré and R. Clouston, *Syntactic Interpolation for Tense Logics and Bi-Intuitionistic Logic via Nested Sequents*, in: M. Fernández and A. Muscholl, editors, *28th EACSL Annual Conference on Computer Science Logic* (*CSL 2020*), Leibniz International Proceedings in Informatics (LIPIcs) **152** (2020), pp. 28:1–28:16.

[3] Nishimura, H., *A study of some tense logics by Gentzen's sequential method*, Publications of the Research Institute for Mathematical Sciences **16** (1980), pp. 343–353.

[4] Ono, H., *Proof-theoretic methods in nonclassical logic –an introduction*, Mathematical Society of Japan Memoirs **2** (1998), pp. 207–254.

[5] Takano, M., *Subformula property as a substitute for cut-elimination in modal propositional logics*, Mathematica Japonica **37** (1992), pp. 1129–1145.

[6] Takano, M., *A modified subformula property for the modal logics K5 and K5D*, Bulletin of the Section of Logic **30** (2001), pp. 115–122.

[7] Takano, M., *A semantical analysis of cut-free calculi for modal logics*, Reports of mathematical logic **53** (2018), pp. 43–65.

A General Completeness Argument for Propositional Dynamic Logics

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Abstract

We formulate a weak completeness argument for Propositional Dynamic Logic that does not rely on the presence of Boolean negation in the language and does not involve a construction of a finite model. As a result, the argument is applicable to a wide range of propositional dynamic logics extending bounded distributive lattice logic, including superintuitionistic and relevant dynamic logics.

Keywords: Completeness, superintuitionistic modal logic, propositional dynamic logic, relevant modal logic, substructural logics.

1 Introduction

It is well-known that, due to non-compactness caused by features of the Kleene star operator on programs, the standard canonical model technique is not applicable in weak completeness proofs for Propositional Dynamic Logic. Instead, known weak completeness proofs [5,3,8,2,1] proceed using a filtration-like construction of a finite counter-model model for each non-theorem. The proof that such a structure invalidates the non-theorem at hand is relying on the fact that sets of states in the model can be characterized by formulas; the proof of this fact usually relies on the presence of Boolean negation. Hence, the standard weak completeness argument unsuitable for generalizations of PDL to logics without Boolean negation or the finite model property. Examples include intuitionistic, paraconsistent or relevant propositional dynamic logics.

In this paper we formulate a generalization of the standard weak completeness argument that avoids these limitations. We show that a version of the modal part of the Segerberg axiomatization of classical PDL is robust in the

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sense that it axiomatizes the PDL program constructs on roughly their expected semantic interpretation independently of the non-modal propositional base, if that base is at least as strong as bounded distributive lattice logic. We supplement the general completeness argument with case studies, including intuitionistic and relevant PDL. The application of our main result to relevant PDL apparently yields the first completeness results for PDL based on $\top\bot$-expansions of some prominent strong relevant logics lacking the finite model property, thus being a step towards solving an open problem pointed out by Sylvan in the early 1990s; see [9]. The results presented here substantially generalize our previous work on the topic [6,7].

2 Preliminaries

Let $\textsc{Fma}$ be a countable set of atomic formulas and $\mathcal{L}$ be any propositional language, comprising a set of operators $\textsc{Op}_{\mathcal{L}}$ together with an arity function $r : \textsc{Op}_{\mathcal{L}} \to \omega$; we usually write o^n to point out that $r(o) = n$ for $o \in \textsc{Op}_{\mathcal{L}}$. It is assumed that $\mathcal{L}$ contains at least the binary operators $\wedge, \vee, \to$, the unary operator $\neg$, and constants $\top, \bot$. Let $\textsc{Pra}$ be a countable set of atomic program expressions. *Programs* and *formulas* of $\mathcal{L}$ are defined as follows:

- $\textsc{Pr}_{\mathcal{L}}$ $\quad P ::= p_i \mid P_0; P_1 \mid P_0 \cup P_1 \mid P^* \mid A?$;
- $\textsc{Fm}_{\mathcal{L}}$ $\quad A ::= a_i \mid o^n(A_0, \dots, A_{n-1}) \mid [P]A$;

where $p_i \in \textsc{Pra}$ and $a_i \in \textsc{Fma}$. Formulas and programs are jointly referred to as *expressions* of $\mathcal{L}$, the set of which is denoted as $\textsc{Ex}_{\mathcal{L}}$. We write PQ instead of $P; Q$ and $A \leftrightarrow B$ instead of $(A \to B) \wedge (B \to A)$.

A *propositional dynamic logic over* $\mathcal{L}$ is any subset of $\textsc{Fm}_{\mathcal{L}}$ that contains all instances of the axioms (officially, we take "axioms" of the form $A \leftrightarrow B$ to represent *pairs* of axioms $A \to B, B \to A$)

A1 $A \to A$	A8 $\bot \to A$
A2 $A \wedge B \to A$	A9 $[P]A \wedge [P]B \to [P](A \wedge B)$
A3 $A \wedge B \to B$	A10 $\top \to [P]\top$
A4 $A \to A \vee B$	A11 $[P \cup Q]A \leftrightarrow ([P]A \wedge [Q]A)$
A5 $B \to A \vee B$	A12 $[P; Q]A \leftrightarrow [P][Q]A$
A6 $A \wedge (B \vee C) \to (A \wedge B) \vee C$	A13 $[P^*]A \to A \wedge [P][P^*]A$
A7 $A \to \top$	A14 $[A?]B \leftrightarrow (\neg A \vee B)$

and is closed under the rules

R1 $\dfrac{A \to B \quad B \to C}{A \to C}$ R3 $\dfrac{A \to C \quad B \to C}{A \vee B \to C}$ R5 $\dfrac{A \to [P]A \wedge B}{A \to [P^*]B}$

R2 $\dfrac{A \to B \quad A \to C}{A \to B \wedge C}$ R4 $\dfrac{A \to B}{[P]A \to [P]B}$

Definition 2.1 An *abstract dynamic model* for $\mathcal{L}$ is $\mathfrak{M} = (K, \leq, O, \Sigma, I)$ such that $K \neq \emptyset$, $\leq$ is a partial order on K, $O \subseteq K$ such that $x \in O$ and $x \leq y$ only if $y \in O$, Σ is an arbitrary set of relations and $I : \mathrm{Ex}_{\mathcal{L}} \to K \times K$ such that ($Id(X) = \{(x,x) \mid x \in X\}$)

(i) $I(A) \subseteq Id(K)$ for all $A \in \mathrm{Fm}_{\mathcal{L}}$

(ii) $(x,x) \in I(A)$ and $x \leq y$ only if $(y,y) \in I(A)$;

(iii) $I(\top) = Id(K)$ and $I(\bot) = \emptyset$;

(iv) $I(A \wedge B) = I(A) \cap I(B)$ and $I(A \vee B) = I(A) \cup I(B)$;

(v) $I([P]A) = \{(x,x) \mid \forall y((x,y) \in I(P) \implies (y,y) \in I(A))\}$;

(vi) $(x,y) \in I(p_i)$ and $z \leq x$ only if $(z,y) \in I(p_i)$;

(vii) $I(P \cup Q) = I(P) \cup I(Q)$;

(viii) $I(PQ) = I(P) \circ I(Q)$;

(ix) $I(P^*) = \big(\leq \cup I(P)\big)^* = \bigcup_{n\in\omega}(\leq \cup I(P))^n$;

(x) $I(A?) = \{(x,y) \mid x \leq y \ \& \ (y,y) \notin I(\neg A)\}$.

We write $x \vDash_{\mathfrak{M}} A$ if $(x,x) \in I(A)$ in $\mathfrak{M}$ and xPy if $(x,y) \in I(P)$. Formula A is *valid in model* $\mathfrak{M}$ iff $Id(O) \subseteq I(A)$.

Note that items (iii–iv) are compatible with the "hereditarity condition" (ii) and items (vi–x) ensure that (v) is compatible with (ii) as well. The relations in Σ (the "signature" of $\mathfrak{M}$) are typically used to specify the interpretation of propositional connectives other than $\top, \bot, \wedge$ and $\vee$, including $\to$ and $\neg$. The precise way how this is done will not matter to us; we do not assume *any* specific properties of $\to, \neg$ other than hereditarity (ii).

Definition 2.2 An abstract dynamic model for $\mathcal{L}$ satisfies the *implication property* if (i) $I(A) \subseteq I(B)$ and (ii) $Id(O) \subseteq I(A \to B)$ are equivalent for all $A, B \in \mathrm{Fm}_{\mathcal{L}}$.

Lemma 2.3 *If $\mathfrak{M}$ for $\mathcal{L}$ satisfies the implication property, then all $\mathcal{L}$-formulas of the form A1–14 are valid in $\mathfrak{M}$ and the set of $\mathcal{L}$-formulas valid in $\mathfrak{M}$ is closed under R1–5.*

3 The main observation

Our main observation is that, for each $A \notin \mathsf{L}$, a structure obtained by combining the canonical model construction and the filtration of the canonical model invalidates A, if the structure satisfies a "readiness" condition that pertains to the behaviour of non-modal formulas. This observation can be used to obtain weak completeness proofs for a wide range of PDLs based on extensions of distributive lattice logic.

Definition 3.1 Let α be a set of $\mathcal{L}$-formulas. The *closure* of α is the smallest set $\alpha' \supseteq \alpha$ that is closed under subformulas and (i) $\top, \bot \in \alpha'$; (ii) if $[P]A \in \alpha'$, then $[P]\top \in \alpha'$; (iii) if $[P;Q]A \in \alpha'$, then $[P][Q]A \in \alpha'$; (iv) if $[P \cup Q]A \in \alpha'$, then $[P]A \in \alpha'$ and $[Q]A \in \alpha'$; (v) if $[P^*]A \in \alpha'$, then $[P][P^*]A \in \alpha'$; (vi) if $[A?]B \in \alpha'$, then $\neg A \in \alpha'$. A set α is *closed* iff α is the closure of α.

Let Φ be an arbitrary set of formulas. A formula A *occurs in* Φ if $A \in \Phi$; a program P occurs in Φ if there is $[P]A$ that occurs in Φ.

Definition 3.2 Let Φ be a finite closed set. A *canonical* L-*model for* Φ is any abstract dynamic model $\mathfrak{M}^{\Phi}_{\mathsf{L}} = (K, \leq, O, \Sigma, I)$ where

(i) K is the set of all prime L-theories and $\leq$ is set inclusion;

(ii) $O = \{\alpha \mid \mathsf{L} \subseteq \alpha\}$;

(iii) $I(a) = \{(\alpha, \alpha) \mid a \in \alpha\}$ for $a \in \text{Fma} \cap \Phi$ and $I(a) = \emptyset$ for $a \in \text{Fma} \setminus \Phi$;

(iv) $I(p) = \{(\alpha, \beta) \mid \forall [p]A \in \Phi([p]A \in \alpha \implies A \in \beta)\}$ for p that occurs in Φ and $I(p) = \emptyset$ if p does not occur in Φ.

Let $\mathcal{C}(\mathfrak{M}^{\Phi}_{\mathsf{L}}) = \{A \in \text{Fm}_{\mathcal{L}} \mid \forall \alpha \in \mathfrak{M}^{\Phi}_{\mathsf{L}}(\alpha \vDash A \iff A \in \alpha)\}$. We write $\mathcal{C}$ instead of $\mathcal{C}(\mathfrak{M}^{\Phi}_{\mathsf{L}})$ if the parameter is clear from the context. Note that $\top, \bot \in \mathcal{C}$ and $A \wedge B, A \vee B \in \mathcal{C}$ for all $A \wedge B, A \vee B \in \Phi$ such that $A, B \in \mathcal{C}$ by definition of prime theory. Our main observation is that, in fact, $\Phi \subseteq \mathcal{C}$, if the propositional part of Φ "behaves as it should", in the sense of the following definition.

Definition 3.3 $\mathfrak{M}^{\Phi}_{\mathsf{L}}$ is *ready* iff

(i) $\text{Fma} \cap \Phi \subseteq \mathcal{C}$; and

(ii) if $B = o^n(A_0, \ldots, A_{n-1}) \in \Phi$ and $A_i \in \mathcal{C}$ for all $i < n$, then $B \in \mathcal{C}$.

For any $X \subseteq K$, we define

- $F_X := \bigvee \{ \bigwedge(\alpha \cap \Phi) \mid \alpha \in X \}$, where $\bigwedge \emptyset := \top$, $\bigvee \emptyset := \bot$, and $F_\alpha := F_{\{\alpha\}}$;
- and, for any $P \in \text{Pr}_{\mathcal{L}}$, let $[P]X := \{\alpha \mid \forall \beta((\alpha, \beta) \in I(P) \implies \beta \in X)\}$.

The following key theorem is a generalizes a result in [4].

Theorem 3.4 *Take any ready* $\mathfrak{M}^{\Phi}_{\mathsf{L}}$ *for any propositional dynamic logic* L *and any finite closed* Φ. *The following hold for all* $E \in \text{Ex}_{\mathcal{L}}$ *occurring in* Φ*:*

(i) *If* $E \in \text{Fm}_{\mathcal{L}}$, *then* $\alpha \vDash E$ *iff* $E \in \alpha$, *for all* $\alpha \in K$*;*

(ii) *If* $E \in \text{Pr}_{\mathcal{L}}$, *then* $[E]A \in \alpha$ *and* $\alpha E \beta$ *only if* $A \in \beta$ *for all* $[E]A \in \Phi$*;*

(iii) *If* $E \in \text{Pr}_{\mathcal{L}}$, *then* $X \subseteq [E]Y$ *only if* $\vdash_{\mathsf{L}} F_X \to [E]F_Y$, *for all* $X, Y \subseteq K$.

The proof is by induction on the complexity of E; the complexity measure uses an elaboration of the "subexpression" relation. We omit the details.

It follows from Theorem 3.4 that if $A \notin \mathsf{L}$ and $\mathfrak{M}$ is a ready canonical L-model for some finite closed set Φ containing A, then A is not valid in $\mathfrak{M}$.

Theorem 3.5 *For all p.d.l.* L *and all classes* $\mathbf{M}$ *of abstract dynamic models, if* $A \notin \mathsf{L}$ *implies that there is a ready canonical* L-*model for some finite closed set* Φ *containing* A *and this canonical* L-*model is in* $\mathbf{M}$, *then* L *is weakly complete with respect to* $\mathbf{M}$.

The presence of $\bot, \top$ is convenient and, so it seems, also necessary for our proof of Theorem 3.4 to go through. In particular, $\bot, \top$ facilitate a general definition of F_X applicable to all $X \subseteq K$ (note that $\alpha \cap \Phi \neq \emptyset$ for all $\alpha \in K$ and F_X is defined even if $X = \emptyset$), and they avoid the problematic case $\{B \mid [p]B \in \alpha \cap \Phi\} = \emptyset$ in the proof of the base case of claim (iii) of Theorem 3.4.

4 Special cases

In order to show that a given propositional dynamic logic L is sound and weakly complete with respect to some class of abstract dynamic models $\mathbf{M}$ using Theorem 3.5, it is sufficient to show that L is sound with respect to $\mathbf{M}$ and that for all $A \notin \mathsf{L}$ there is a ready $\mathfrak{M}^{\Phi}_{\mathsf{L}}$ for $A \in \Phi$ such that $\mathfrak{M}^{\Phi}_{\mathsf{L}} \in \mathbf{M}$. Readiness and $\mathfrak{M}^{\Phi}_{\mathsf{L}} \in \mathbf{M}$ can usually be established using well-known facts about prime L_0-theories, where L_0 is the non-modal fragment of L; soundness follows from Lemma 2.3 and properties of non-modal prime theories. Specific L to which this sort of argument applies include classical PDL, a version of intuitionistic PDL, a wide range of canonical superintuitionistic PDLs, and canonical relevant PDLs extending the basic relevant logic B, including PDLs based on the prominent relevant logics T, R and E; these logics are known to lack the finite model property [10] (in fact, they are undecidable). The latter result is a step towards the solution of a problem left open by Sylvan [9]; the reason for reservation here is that we are using constants $\top, \bot$ which are problematic from the relevantist standpoint (one reason is that they violate the variable sharing property; another reason is that some relevantists argue that there is no proposition that is true, or one that is false, in all states). Hence, a natural open problem is to replicate our argument without using $\top$ and $\bot$.

References

[1] Harel, D., D. Kozen and J. Tiuryn, "Dynamic Logic," MIT Press, 2000.

[2] Kozen, D. and R. Parikh, *An elementary proof of the completeness of PDL*, Theoretical Computer Science **14** (1981), pp. 113–118.

[3] Nishimura, H., *Sequential method in propositional dynamic logic*, Acta Informatica **12** (1979), pp. 377–400.

[4] Nishimura, H., *Semantical analysis of constructive PDL*, Publications of the Research Institute for Mathematical Sciences **18** (1982), pp. 847–858.

[5] Parikh, R., *The completeness of propositional dynamic logic*, in: J. Winkowski, editor, *Mathematical Foundations of Computer Science 1978* (1978), pp. 403–415.

[6] Punčochář, V. and I. Sedlár, *From positive PDL to its non-classical extensions*, Logic Journal of the IGPL **27** (2019), pp. 522–542.

[7] Sedlár, I., *Substructural propositional dynamic logics*, in: R. Iemhoff, M. Moortgat, R. de Queiroz, editors, *WoLLIC 2019* (2019), pp. 594–609.

[8] Segerberg, K., *A completeness theorem in the modal logic of programs*, Banach Center Publications **9** (1982), pp. 31–46.

[9] Sylvan, R., *Process and action: Relevant theory and logics*, Studia Logica **51** (1992), pp. 379–437.

[10] Urquhart, A., *The undecidability of entailment and relevant implication*, The Journal of Symbolic Logic **49** (1984), pp. 1059–1073.

Some prospects for semiproducts and products of modal logics

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Abstract

We consider products and semiproducts of propositional modal logics Λ with **S5** and present new examples of product and semiproduct logics axiomatized in the 'minimal' way and enjoying the product (or semiproduct) FMP. An essential part of the proof is local tabularity of these (semi)products for Λ of finite depth; it is obtained by using bisimulation games. These results readily imply decidability for 1-variable fragments of predicate modal logics $\mathbf{Q}\Lambda$ and $\mathbf{Q}\Lambda$+Barcan formula. We also present new counterexamples, i.e. (semi)products not axiomatizable in the simplest way.

Keywords: modal logic, 1-variable fragment, product of modal logics, bisimulation game, finite model property

1 Introduction

Semiproducts and products are special types of combined modal logics. Their systematic investigation began in the 1990s, notably due to connections with other areas of logic, both pure and applied, cf. [2]. Nowadays the field has become even more interesting and intriguing; for an overview of some developments cf. [6]. In this note we are especially interested in (semi)products with **S5**, due to their interpretation in modal predicate logic translating the **S5**-necessity into the universal quantifier.

One of the starting points in the study of products was the "product-matching" theorem ([2], Theorem 5.9) — the product of two Kripke complete Horn axiomatizable logics is axiomatized in the minimal way. A similar result for semiproducts ("semiproduct-matching") is known for particular cases only (ibid., Theorem 9.10). Here we present some new positive examples — Horn axiomatizable logics that are semiproduct-matching with **S5** and have the product finite model property (FMP). This implies decidability and the FMP for corresponding 1-variable modal predicate logics.

We also present new counterexamples — two infinite families of logics not semiproduct-matching with **S5**. In particular, we show that Horn axiomatizable complete logics may not be semiproduct-matching.

2 Preliminaries

We consider normal monomodal predicate logics, as defined in [4], in a signature with predicate letters only. A logic is a set of formulas containing standard first-order axioms and the axiom of **K** and closed under standard rules (including predicate substitution). The minimal predicate extension of a propositional monomodal logic Λ is denoted by $\mathbf{Q}\Lambda$; $\mathbf{Q}\Lambda\mathbf{C}$ denotes $\mathbf{Q}\Lambda + \forall x \Box P(x) \to \Box \forall x\, P(x)$ (the Barcan axiom).

Formulas constructed from a single variable x and monadic predicate letters are called *1-variable*. Formulas in which every subformula of the form $\Box B$ contains at most one parameter are called *monodic* [2].

Lemma 2.1 *Every monadic monodic formula with at most one parameter is equivalent to a 1-variable formula in* **QK**.

In turn, every monomodal 1-variable formula A translates into a bimodal propositional formula A_* with modalities $\Box$ and $\blacksquare$, if every atom $P_i(x)$ is replaced with a proposition letter p_i and every quantifier $\forall x$ with $\blacksquare$. The *1-variable fragment* of a predicate logic L is the set

$$L{-}1 := \{A_* \mid A \in L,\ A \text{ is 1-variable}\}.$$

For a modal predicate logic L, we have the following:

Lemma 2.2 $L{-}1$ *is a bimodal propositional logic containing* $\mathbf{K} \rfloor \mathbf{S5}$.

Definition 2.3 The *product* of frames $F_1 = (U_1, R_1),\ F_2 = (U_2, R_2)$ is $F_1 \times F_2 := (U_1 \times U_2, R_h, R_v)$, where

$$R_h(u, v) = R_1(u) \times \{v\},\ R_v(u, v) = \{u\} \times R_2(v).$$

A *semiproduct* of F_1 and F_2 is a subframe $(F_1 \times F_2)|W$ where $R_h(W) \subseteq W$.

Consider a monomodal propositional logic Λ (in the language with $\Box$) and **S5** (in the language with $\blacksquare$). Put

$$\Lambda \rfloor \mathbf{S5} := \Lambda * \mathbf{S5} + \Box\blacksquare p \to \blacksquare\Box p, \quad [\Lambda, \mathbf{S5}] := \Lambda \rfloor \mathbf{S5} + \blacksquare\Box p \to \Box\blacksquare p,$$

where $*$ denotes fusion.

Definition 2.4 The *product* $\Lambda \times \mathbf{S5}$ is the logic of the class of all products of Λ-frames with **S5**-frames. Similarly, the *semiproduct* $\Lambda \leftthreetimes \mathbf{S5}$ is the logic of the class of all semiproducts of such frames.

In both cases, instead of arbitary **S5**-frames one can use single clusters.

Definition 2.5 The *Kripke-completion* $\overline{L}$ of a modal predicate logic L is the logic of the class of all predicate Kripke frames validating L.

Lemma 2.6 (i) $\Lambda \lrcorner\, \mathbf{S5} \subseteq \mathbf{Q}\Lambda{-}1 \subseteq \overline{\mathbf{Q}\Lambda}{-}1 = \Lambda \leftthreetimes \mathbf{S5}$.

(ii) $[\Lambda, \mathbf{S5}] \subseteq \mathbf{Q}\Lambda\mathbf{C}{-}1 \subseteq \overline{\mathbf{Q}\Lambda\mathbf{C}}{-}1 = \Lambda \times \mathbf{S5}$.

Definition 2.7 Λ and $\mathbf{S5}$ are called *semiproduct-matching* if $\Lambda \lrcorner\, \mathbf{S5} = \Lambda \leftthreetimes \mathbf{S5}$ and *product-matching* if $[\Lambda, \mathbf{S5}] = \Lambda \times \mathbf{S5}$.

Λ is called *quantifier-friendly*, if $\mathbf{Q}\Lambda - 1 = \Lambda \lrcorner\, \mathbf{S5}$, and *Barcan-friendly*, if $\mathbf{Q}\Lambda\mathbf{C}{-}1 = \Lambda \times \mathbf{S5}$.

So Λ is quantifier-friendly (respectively, Barcan-friendly) whenever Λ and $\mathbf{S5}$ are semiproduct-matching (respectively, product-matching).

Theorem 2.8 *(cf. [2], Theorem 5.9). If Λ is Kripke complete and Horn axiomatizable, then Λ and $\mathbf{S5}$ are product-matching.*

For semiproducts an analogue of this theorem does not hold (see below). Let us recall, in a slightly more general form, a number of positive results presented in [2], Theorem 9.10. [1]

Definition 2.9 A *one-way PTC-logic* is a modal propositional logic axiomatized by formulas of the form $\Box p \to \Box^n p$ and variable-free formulas.

Theorem 2.10 *Λ and $\mathbf{S5}$ are semiproduct-matching for any one-way PTC-logic Λ.*

3 Counterexamples

Theorem 3.1 *(cf. [9]) Let*

$$\Box\mathbf{T} := \mathbf{K} + \Box(\Box p \to p), \quad \mathbf{SL4} := \mathbf{K4} + \Diamond p \leftrightarrow \Box p.$$

If $\Box\mathbf{T} \subseteq \Lambda \subseteq \mathbf{SL4}$, then Λ and $\mathbf{S5}$ are not semiproduct-matching.

For the proof note that $\Box\blacksquare(\Box p \to p) \in (\Lambda \leftthreetimes \mathbf{S5}) - (\Lambda \lrcorner\, \mathbf{S5})$.

Hence we obtain counterexamples to an analogue of Theorem 2.8: Horn axiomatizable logics $\Box\mathbf{T}$, $\mathbf{K5}$, $\mathbf{K45}$ are not semiproduct-matching with $\mathbf{S5}$.

Nevertheless, we have

Remark 3.2 (cf. [8]) *Every complete Horn axiomatizable logic is quantifier-friendly.*

Theorem 3.3 *If $\mathbf{K} + Alt_n \subseteq \Lambda \subseteq \mathbf{K} + Alt_n + \Box^m\bot$ for $n \geq 3$, $m \geq 2$, then Λ and $\mathbf{S5}$ are neither product- nor semiproduct-matching.*

Proof. (Sketch.) Take the product $F_1 \times F_2$, where F_1 is the irreflexive tree with the root 0 and the leaves $1, \ldots, n$ and F_2 is the two-element cluster $\{1, 2\}$; replace R_v by the least equivalence relation S_2 such $(x, y)S_2(x', y')$ for $x = x' = 0$ or $x = x' > 3$, $(1, 1)S_2(2, 2)$, $(1, 1)S_2(3, 2)$, $(1, 2)S_2(2, 1)$, $(1, 2)S_2(3, 1)$. The

[1] In [2] semiproducts are called 'expanding relativized products', $\Lambda \lrcorner\, \mathbf{S5}$ is denoted by $[\Lambda, \mathbf{S5}]^{EX}$, $\Lambda \leftthreetimes \mathbf{S5}$ by $(\Lambda \times \mathbf{S5})^{EX}$.

resulting frame G_n is not a p-morphic image of a semiproduct of a $(\mathbf{K}+Alt_n)$-frame and a cluster while $G_n \vDash [\mathbf{K}+Alt_n+\Box^2\bot, \mathbf{S5}]$. Therefore its Fine-Jankov formula belongs to $(\Lambda \leftthreetimes \mathbf{S5}) - [\Lambda, \mathbf{S5}]$. ■

A standard canonical model argument proves Kripke-completeness of all the logics $\mathbf{Q}\Lambda$ for $\Lambda = \mathbf{K} + Alt_n,\ \mathbf{K} + Alt_n + \Box^m\bot$. So we obtain

Corollary 3.4 *The logics* $\mathbf{K}+Alt_n$, $\mathbf{K}+Alt_n+\Box^m\bot$ *are not quantifier-friendly for* $n \geq 3$, $m \geq 2$.

4 Local tabularity

Recall that a propositional logic L is *locally tabular*, if for any finite k there exist finitely many L-non-equivalent formulas in k proposition letters.

It is well known that every extension of a locally tabular modal logic in the same language is locally tabular; every locally tabular logic has the FMP.

Theorem 4.1 *Every logic* $(\mathbf{K}+\Box^n\bot) \lrcorner\, \mathbf{S5}$ *is locally tabular.*

This theorem is proved by using bisimulation games; the corresponding technique is described in [7].

A monomodal logic Λ is *of finite depth* if $\Box^n\bot \in \Lambda$ for some n.

Corollary 4.2 *If* Λ *is of finite depth, then the logics* $\Lambda \leftthreetimes \mathbf{S5}$, $\Lambda \lrcorner\, \mathbf{S5}$ *have the FMP; so their finite axiomatizability implies decidability.*

In particular, $\Lambda \leftthreetimes \mathbf{S5}$ $(\Lambda \times \mathbf{S5})$ is decidable, provided Λ, $\mathbf{S5}$ are semiproduct-(product-) matching and Λ is of finite depth.

5 More examples of semiproduct-matching

In contrast with Theorem 3.3, we can identify some other logics that are semiproduct-matching with **S5**.

Lemma 5.1 *Consider the axiom* $Ath := \Diamond\Diamond p \to \Box\Diamond p$. *Ath-frames are defined by the following first-order condition:*

$$\forall x, y, z, u\, (xRy \wedge xRz \wedge yRu \to zRu).$$

We call these frames *thick*.

Proposition 5.2 *The logics* $\mathbf{K} + Ath$, $\mathbf{K} + Ath + \Box^n\bot$ *for* $n \geq 1$ *are semiproduct-matching with* **S5**.

Proof. (Sketch.) Every countable rooted $\mathbf{K} \lrcorner\, \mathbf{S5}$-frame H is a p-morphic image of a semiproduct G of a tree F and a cluster C; the proof is similar to the one for products, cf. [2]. Since Ath is a Horn formula, we can take the corresponding Horn closure G^+; then G^+ is a semiproduct of F^+ and C. If $H \vDash Ath$, we obtain a p-morphism from G^+ onto H. So every formula refutable on H is not in $(\mathbf{K}+Ath) \leftthreetimes \mathbf{S5}$.

Adding variable-free axioms $\Box^n\bot$ does not affect this argument. ■

6 Product and semiproduct FMP

In many cases (semi)products enjoy the (semi)product FMP. In particular, if L_1 is tabular and L_2 has the FMP, then $L_1 \times L_2$ has the product FMP [3, Cor. 5.9]. Probably, this may not be true, if L_1 is only locally tabular. Examples of semiproduct FMP can be found in [5], but they do not cover our next result:

Theorem 6.1 *For $\Lambda = \mathbf{K} + Ath$ and $\Lambda = \mathbf{K} + \Box^n\bot + Ath$, the (semi)product of Λ with* **S5** *has the (semi)product FMP.*

Corollary 6.2 *For logics Λ from Theorem 6.1 $\mathbf{Q}\Lambda - 1$ has the FMP, i.e., is complete w.r.t. finite Kripke frames with finite domains.*

Let us give some comments about the proof of Theorem 6.1 for the case of semiproducts. Note that $(\mathbf{K} + Ath) \leftthreetimes \mathbf{S5} = \bigcap_n((\mathbf{K} + \Box^n\bot + Ath) \leftthreetimes \mathbf{S5})$, so it suffices to consider only $L = \Lambda \leftthreetimes \mathbf{S5}$ for $\Lambda = \mathbf{K} + \Box^n\bot + Ath$ and show that every finite rooted L-frame $F = (W, R_1, R_2)$ is a p-morphic image of a finite semiproduct of a Λ-frame with a cluster. A *row* in F is a connected component in (W, R_1); a *column* is an equivalence class under R_2; a *block* is a non-empty intersection of a row and a column. F is *straight* if all its blocks are singletons. We can show that F is a p-morphic image of a straight rooted L-frame isomorphic to a semiproduct of a Λ-frame and a cluster.

Remark 6.3 We hope our main results can be transferred to extensions of **GL**. The logic $\mathbf{GL} \rfloor \mathbf{S5}$ is the well-known provability logic of Artemov–Japaridze, which is semiproduct-matching with **S5**. A transitive analogue of *Ath* is R. Solovay's axiom $AS := \Box(\Box p \to \Box q) \vee \Box(\Box q \to p \wedge \Box p)$. We may conjecture that $\mathbf{SOL} := \mathbf{GL} + AS$ (Solovay's logic of "provability w.r.t **ZF**" cf. [1], ch. 13) is also semiproduct-matching with **S5** and that $\mathbf{SOL} \leftthreetimes \mathbf{S5}$ has the semiproduct FMP.

References

[1] Boolos, G., *The logic of provability.* Cambridge University Press, 1993.

[2] Gabbay, D., A. Kurucz, F. Wolter and M. Zakharyaschev, "Many-dimensional Modal Logics: Theory and Applications," Elsevier, 2003.

[3] Gabbay, D., I. Shapirovsky and V. Shehtman, *Products of modal logics and tensor products of modal algebras*, Journal of Applied Logic **12** (2014), pp. 570–583.

[4] Gabbay, D., V. Shehtman and D. Skvortsov, "Quantification in Nonclassical Logic, Volume 1," Studies in Logic and the Foundations of Mathematics **153**, Elsevier, 2009.

[5] Gabelaia, D., A. Kurucz, F. Wolter and M. Zakharyaschev, *Non-primitive recursive decidability of products of modal logics with expanding domains*, Annals of Pure and Applied Logic **142** (2006), pp. 245–268.

[6] Kurucz, A., *Combining modal logics*, in: P. Blackburn, J. V. Benthem and F. Wolter, editors, *Handbook of Modal Logic*, Elsevier, 2008 pp. 869–924.

[7] Shehtman, V., *Segerberg squares of modal logics and theories of relation algebras*, in: S. Odintsov, editor, *Larisa Maksimova on Implication, Interpolation, and Definability*, Springer, 2018 pp. 245–296.

[8] Shehtman, V., *Simplicial semantics and one-variable fragments of modal predicate logics*, in: *TACL 2019* (2019), pp. 172–173.

[9] Shehtman, V. and D. Shkatov, *On one-variable fragments of modal predicate logics*, in: *SYSMICS 2019, Booklet of Abstracts* (2019), pp. 129–132.

Knowledge-based Conditional Obligation

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Abstract

Obligations for an agent may depend on its knowledge. In order to formalize knowledge-based obligations, we present the logic $\mathbb{KCDL}$ (Knowledge-based Conditional Deontic Logic) based on Hansson's style of conditional obligations, incorporating epistemic information. $\mathbb{KCDL}$ is based on a new dyadic operator, called epistemic conditional obligation. The complete axiomatization of the logic is given as well.

1 Introduction

Obligations of agents can be affected by their knowledge. For example, a doctor should not be blamed for not treating a man when she does not know to be ill, although the doctor bears an objective obligation to treat patients. This paper focuses on knowledge-based obligation [5] and describes it with a new dyadic obligation operator from a view of conditional obligations. The dyadic operator $\bigcirc(\phi|\psi)$ is read as: it ought to be ϕ given the condition that ψ [8]. Hansson proposed a new dyadic obligation operator over preference-based models [3], where the semantics of $\bigcirc(\phi|\psi)$ is: on the best ψ-states, ϕ is satisfied.

In this paper, we intend to formalize those conditional obligations that the agent already knows, but the knowledge of the antecedents decides whether the conditional obligations are 'triggered'. As the above example shows, the doctor knows that she is obliged to treat the man under the condition that he is ill, but she does not know whether the man is ill. We will follow Hansson's method to define a new dyadic deontic operator $\bigodot(\phi|\psi)$, called *epistemic conditional obligations* based on *epistemic betterness structures* where epistemic relations are introduced.

2 Language and Epistemic Betterness Structures

2.1 The Language for $\mathbb{KCDL}$

Definition 2.1 (Language $\mathcal{L}_{KCDL}$) Let P be a set of propositional variables. The language $\mathcal{L}_{KCDL}$ is given by the following BNF:

$$\phi ::= p \mid \neg\phi \mid (\phi \wedge \phi) \mid K\phi \mid \bigodot(\phi|\phi)$$

$K\phi$ represents knowledge. $\bigodot(\phi|\psi)$ represents the epistemic conditional obligation, which can be read as: the agent knows that over all the cases considered possible, it ought to be ϕ given the condition that ψ.

2.2 Semantics of $\mathbb{KCDL}$

For the semantics we need epistemic betterness structures.

Definition 2.2 (Epistemic Betterness Structures) $M = \langle S, \sim, \leqslant, V\rangle$ is an epistemic betterness structure where S is the set of states, $\sim$: $S \times S$ is an epistemic relation (equivalence relation), $\leqslant$: $S \times S$ is a partial order, called betterness relation and $V : P \to \mathcal{P}(S)$ is the valuation over S. Let $[s]^{\sim}$ be the set of states accessible from s by the epistemic relation $\sim$. Let $\|\phi\|_M$ be the set of all the states satisfying ϕ in M. $s > t$ if $t \leqslant s$ and $s \nleqslant t$.

Accordingly, maximality and truth conditions can be defined as follows:

Definition 2.3 Given an arbitrary epistemic betterness structure $M = \langle S, \sim, \leqslant, V\rangle$: $r \in \max_{\leqslant} \|\phi\|_M \Leftrightarrow r \in \|\phi\|_M$ and $\forall t \in \|\phi\|_M (r \leqslant t \Rightarrow t \leqslant r)$, $r \in \max_{\leqslant|[s]^{\sim}} \|\phi\|_M \Leftrightarrow r \in \|\phi\|_M \cap [s]^{\sim}$ and $\forall t \in \|\phi\|_M \cap [s]^{\sim} (r \leqslant t \Rightarrow t \leqslant r)$. The truth condition of $\bigodot(\phi|\psi)$ can be defined over M as: $M, s \models \bigodot(\phi|\psi) \Leftrightarrow \max_{\leqslant|[s]^{\sim}} \|\psi\|_M \subseteq \|\phi\|_M$.

Intuitively, no other element is strictly better than any maximal element of a partially ordered set S. The truth condition of $\bigodot(\phi|\psi)$ means over all the states that are indistinguishable from s, the best ψ-states also satisfy ϕ.

To make the semantics work, we need to assume that $\leqslant$ should be $\sim$-smooth, which is inspired by the notion of smoothness in Parent's work [6].

Definition 2.4 ($\sim$-Smoothness) An epistemic betterness structure M is $\sim$-smooth if for every state s in M, for every $t \in [s]^{\sim}$, if $M, t \models \phi$, either $t \in \max_{\leqslant|[s]^{\sim}} \|\phi\|_M$ or $\exists v \in [s]^{\sim} : v > t$ *and* $v \in \max_{\leqslant|[s]^{\sim}} \|\phi\|_M$.

2.3 Epistemic Factual Detachment

In the tradition of Hansson's framework, factual detachment can be formalized as $\bigcirc(\phi|\psi) \wedge \Box\psi \to \bigcirc(\phi|\top)$ (see [7]), which describes the detachment of the antecedent in the conditional obligation due to its necessity.

In our framework, we can formalize an epistemic version of factual detachment based on epistemic conditional obligations as: $(\bigodot(\phi|\psi) \wedge K\psi) \to \bigodot(\phi|\top)$ (EFD). (EFD) is valid over epistemic betterness structures. It is in line with our intuitions: knowing the antecedent triggers the conditional obligation, making the consequence unconditionally obligatory.

3 Logic of Knowledge-based Conditional Obligation

3.1 Axiom System for $\mathbb{KCDL}$

TAUT	(PL)
$\mathcal{S}5$-schema for K	($\mathcal{S}5$)
$\bigodot(\psi \to \gamma|\phi) \to (\bigodot(\psi|\phi) \to \bigodot(\gamma|\phi))$	($\bigodot$K)
$\bigodot(\psi|\phi) \to K\bigodot(\psi|\phi)$	($\bigodot$Abs)
$K\phi \to \bigodot(\phi|\psi)$	($\bigodot$Nec)
$K(\phi \leftrightarrow \psi) \to (\bigodot(\gamma|\phi) \leftrightarrow \bigodot(\gamma|\psi))$	($\bigodot$Ext)
$\bigodot(\phi|\phi)$	($\bigodot$Id)
$\bigodot(\gamma|\phi \wedge \psi) \to \bigodot(\psi \to \gamma|\phi)$	($\bigodot$Sh)

$$\neg K\neg\phi \to (\bigodot(\psi|\phi) \to \neg\bigodot(\neg\psi|\phi)) \qquad (\bigodot\ \mathrm{D}^\star)$$
$$(\bigodot(\psi|\phi) \wedge \bigodot(\gamma|\phi)) \to \bigodot(\gamma|\phi\wedge\psi) \qquad (\bigodot\mathrm{CM})$$
$$\text{If} \vdash \phi \text{ and} \vdash \phi \to \psi, \text{ then} \vdash \psi \qquad (\mathrm{MP})$$
$$\text{If} \vdash \phi, \text{ then} \vdash K\phi \qquad (\mathrm{KN})$$

$\mathbb{KCDL}$ is the same as the system $\mathbf{F}$+(CM) in Parent's paper [6] following Åqvist's approach [1]. But $\mathbf{F}$+(CM) is investigated over reflexive and smooth betterness structures (or reflexive, total [1], transitive and smooth structures).

Theorem 3.1 *(Soundness)* $\mathbb{KCDL}$ *is sound with respect to the class of epistemic betterness structures where* $\leqslant$ *is reflexive, transitive and* $\sim$*-smooth.*

We omit proofs since they are almost the same as Parent's proof.

Lemma 3.2 *The following formulas are derivable in* $\mathbb{KCDL}$*:*

(i) $\bigodot(\psi_1|\phi) \wedge \bigodot(\psi_2|\phi) \wedge \cdots \bigodot(\psi_n|\phi) \to \bigodot(\psi_1 \wedge \psi_2 \cdots \wedge \psi_n|\phi)$ $(n \geqslant 2)$

(ii) *If* $\vdash \psi \to \gamma$*, then* $\vdash \bigodot(\psi|\phi) \to \bigodot(\gamma|\phi)$.

(iii) $\bigodot(\phi|\phi\vee\psi) \wedge \bigodot(\psi|\psi\vee\gamma) \to \bigodot(\gamma\to\psi|\phi)$

(iv) $\neg K\neg\phi \to \neg\bigodot(\bot|\phi)$

(v) $\bigodot(\gamma|\phi\vee\gamma) \wedge \bigodot(\psi|\phi) \to \bigodot(\phi\to\psi|\gamma)$

(vi) $(\bigodot(\phi|\phi\vee\psi) \wedge \bigodot(\psi|\psi\vee\gamma)) \to \bigodot(\phi|\phi\vee\gamma)$

(i) - (v) are proved in [6]. The derivability of (vi) refers to [4].

3.2 Strong Completeness of $\mathbb{KCDL}$

The basic strategy of proving completeness is also attributed to Parent's work. We give a new definition on $\leqslant$ in the canonical models which keeps $\leqslant$ transitive. Let Γ be a consistent set of $\mathcal{L}_{KCDL}$-formulas. We need to establish a canonical model which satisfies Γ. Let Γ_0 be some maximal consistent extension of Γ. Γ_0^ψ denotes $\{\phi \mid \bigodot(\phi|\psi) \in \Gamma_0\}$ and $K^{-1}\Delta$ denotes $\{\phi \mid K\phi \in \Delta\}$. We will distinguish two cases: (1) Principal case: there is a formula ω such that $\Gamma_0^\omega \subseteq \Gamma_0$; (2) Limiting case: there is no formula ω such that $\Gamma_0^\omega \subseteq \Gamma_0$.

3.2.1 Principal Case

Definition 3.3 (The Canonical Model Generated by Γ_0, Principal Case) A canonical model generated by Γ_0 is a tuple $M^{\Gamma_0} = \langle W, \sim, \leqslant, V\rangle$ where

(i) $W = \{(\Delta,\psi) \mid \Delta \textit{ is a } MCS \textit{ and } \Gamma_0^\psi \subseteq \Delta\}$ [2];

(ii) $(\Delta,\psi) \sim (\Sigma,\chi)$ iff $K^{-1}\Delta \subseteq \Sigma$;

(iii) $(\Delta,\psi) \leqslant (\Sigma,\chi)$ iff $(\bigodot(\chi|\chi\vee\psi) \in \Gamma_0$ and $\psi \notin \Sigma)$ or $(\Delta = \Sigma$ and $\psi = \chi)$.

(iv) $V(p) = \{(\Delta,\psi) \mid p \in \Delta\}$ for any $p \in \mathbf{P}$.

Lemma 3.4 *(1)* $\sim$ *is an equivalence relation and total; (2) Let* Δ *be a MCS. If* $\bigodot(\phi|\phi\vee\psi) \notin \Delta$*, then* $\Delta^{\phi\vee\psi} \cup \{\neg\phi\}$ *is consistent; (3) Let* Δ *and* Δ_1 *be two MCSs. If* $\bigodot(\phi|\psi) \notin \Delta_1$ *and* $K^{-1}\Delta \subseteq \Delta_1$*, then* $\Delta^\psi \cup \{\neg\phi\}$ *is consistent.*

---

[1] An order $\leqslant$ is total over a set S iff for any $t_1, t_2 \in S$, $t_1 \leqslant t_2$ or $t_2 \leqslant t_1$.

[2] MCS represents the maximal $\mathcal{L}_{KCDL}$-consistent set.

Now we can prove the Truth Lemma based on M^{Γ_0}.

Lemma 3.5 *(Truth Lemma) Let* $M^{\Gamma_0} = \langle W, \sim, \leqslant, V\rangle$ *be a canonical model generated by* Γ_0*. For all* $(\Delta,\psi) \in W$ *and all* ϕ*,* $M^{\Gamma_0}, (\Delta,\psi) \models \phi$ *iff* $\phi \in \Delta$*.*

Proof. We prove it by induction on the structure of ϕ. When ϕ is a Boolean formula, the proof is standard. When $\phi = K\beta$, it is almost the same as [6].

When $\phi = \bigodot(\alpha|\beta)$:

- $(\Rightarrow)$ Suppose that $\bigodot(\alpha|\beta) \notin \Delta$. By Lemma 3.4(3), $\Gamma_0^\beta \cup \{\neg\alpha\}$ is consistent. So $\Gamma_0^\beta \cup \{\neg\alpha\}$ can be extended into a MCS Δ_1. Since $\Gamma_0^\beta \subseteq \Delta_1$, $(\Delta_1,\beta) \in W$. Let (Δ_2,γ) be an arbitrary state in W such that $\beta \in \Delta_2$. By Definition 3.3, $(\Delta_2,\gamma) \not> (\Delta_1,\beta)$. By Lemma 3.4(1), $(\Delta_1,\beta) \sim (\Delta,\psi)$. Thus, $(\Delta_1,\beta) \in \max_{\leqslant|[(\Delta,\psi)]^\sim} \|\beta\|_{M^{\Gamma_0}}$. By the inductive hypothesis, $M^{\Gamma_0}, (\Delta_1,\beta) \models \neg\alpha$. So $M^{\Gamma_0}, (\Delta,\psi) \not\models \bigodot(\alpha|\beta)$.
- $(\Leftarrow)$ Suppose that $\bigodot(\alpha|\beta) \in \Delta$. Let $(\Delta_1,\theta) \in \max_{\leqslant|[(\Delta,\psi)]^\sim} \|\beta\|_{M^{\Gamma_0}}$. We want to show that $\bigodot(\theta|\beta\vee\theta) \in \Gamma_0$. Assume, to reach a contradiction, that $\bigodot(\theta|\beta\vee\theta) \notin \Gamma_0$. By Lemma 3.4(2), $\Gamma_0^{\beta\vee\theta} \cup \{\neg\theta\}$ is consistent. So it can be extended into a MCS Δ_2 such that $\Gamma_0^{\beta\vee\theta} \cup \{\neg\theta\} \subseteq \Delta_2$. So $(\Delta_2, \beta\vee\theta) \in W$. By the axiom ($\bigodot$Id), $\beta\vee\theta \in \Delta_2$. So $\beta \in \Delta_2$. By ($\bigodot$Id) again, we have $\bigodot(\beta\vee\theta|\beta\vee\theta\vee\theta) \in \Gamma_0$. Since $\theta \notin \Delta_2$, $(\Delta_1,\theta) \leqslant (\Delta_2,\beta\vee\theta)$. And we know $\bigodot(\theta|\beta\vee\theta\vee\theta) \notin \Gamma_0$. So $(\Delta_2,\beta\vee\theta) \not\leqslant (\Delta_1,\theta)$. Thus, $(\Delta_1,\theta) < (\Delta_2,\beta\vee\theta)$. By Lemma 3.4(1), $(\Delta_1,\theta) \sim (\Delta_2,\beta\vee\theta)$. By the inductive hypothesis, $M^{\Gamma_0}, (\Delta_2,\beta\vee\theta) \models \beta$, which contradicts $(\Delta_1,\theta) \in \max_{\leqslant|[(\Delta,\psi)]^\sim} \|\beta\|_{M^{\Gamma_0}}$. Thus, $\bigodot(\theta|\beta\vee\theta) \in \Gamma_0$. Let γ be an arbitrary formula such that $\gamma \in \Gamma_o^\beta$. So $\bigodot(\gamma|\beta) \in \Gamma_0$. We also have $\bigodot(\theta|\beta\vee\theta) \in \Gamma_0$. Thus, by Lemma 3.2(v), $\bigodot(\beta\to\gamma|\theta) \in \Gamma_0$. Thus, $\beta\to\gamma \in \Gamma_0^\theta$. So $\beta\to\gamma \in \Delta_1$. Thus, $\gamma \in \Delta_1$. So $\alpha \in \Delta_1$ as well. Therefore, $M^{\Gamma_0}, (\Delta,\psi) \models \bigodot(\alpha|\beta)$.

□

Lemma 3.6 *(Verification Lemma)* M^{Γ_0} *is reflexive, transitive and* $\sim$*-smooth.*

Proof. (Reflexivity and Transitivity) Reflexivity is easily verified by Definition 3.3. Transitivity can be obtained by Lemma 3.2(iii) and Lemma 3.2(vi).

($\sim$-smoothness) Let $(\Delta,\theta) \in M^{\Gamma_0}$ such that $M^{\Gamma_0}, (\Delta,\theta) \models \beta$:

- When $\bigodot(\theta|\theta\vee\beta) \in \Gamma_0$: Assume that $(\Delta,\theta) \notin \max_{\leqslant|[(\Delta,\theta)]^\sim} \|\beta\|_{M^{\Gamma_0}}$. This means that there exists $(\Sigma,\lambda) \in M^{\Gamma_0}$ such that $(\Sigma,\lambda) > (\Delta,\theta)$ and $\Sigma \in \|\beta\|_{M_0^\Gamma}$. By Definition 3.3(iii), $\bigodot(\lambda|\lambda\vee\theta) \in \Gamma_0$ and $\theta \notin \Sigma$. By Lemma 3.2(v), $\bigodot(\lambda|\lambda\vee\theta) \wedge \bigodot(\theta|\theta\vee\beta) \to \bigodot(\beta\to\theta|\lambda) \in \Gamma_0$. So $\bigodot(\beta\to\theta|\lambda) \in \Gamma_0$. So $\beta\to\theta \in \Sigma$, which implies that $\theta \in \Sigma$. Contradiction.
- When $\bigodot(\theta|\theta\vee\beta) \notin \Gamma_0$, we will show that there is $(\Sigma,\beta\vee\theta) \in M^{\Gamma_0}$ such that $(\Sigma,\beta\vee\theta) > (\Delta,\beta)$ and $(\Sigma,\beta\vee\theta) \in \max_{\leqslant|[(\Delta,\theta)]^\sim} \|\beta\|_{M^{\Gamma_0}}$. Since $\bigodot(\theta|\theta\vee\beta) \notin \Gamma_0$, by Lemma 3.4(2), $\Gamma_0^{\beta\vee\theta} \cup \{\neg\theta\}$ is consistent. So it can be extended into a MCS Σ such that $\Gamma_0^{\beta\vee\theta} \cup \{\neg\theta\} \subseteq \Sigma$. By Definition 3.3, $(\Sigma,\beta\vee\theta) \in M^{\Gamma_0}$. Since $\neg\theta \in \Sigma$, we have $\beta \in \Sigma$. Since for any $(\Lambda,\lambda) \geqslant (\Sigma,\beta\vee\theta)$, $\neg(\beta\vee\theta) \in \Lambda$. So $\neg\beta \in \Lambda$. Thus, $(\Sigma,\beta\vee\theta) \in \max_{\leqslant|[(\Delta,\theta)]^\sim} \|\beta\|_{M^{\Gamma_0}}$. By the axiom ($\bigodot$Id), we have $(\Sigma,\beta\vee\theta) > (\Delta,\beta)$.

□

3.2.2 Limiting Case

Definition 3.7 (The Canonical Model Generated by (Γ_0, ω), Limiting Case) Take an arbitrary formula ω, the canonical model generated by (Γ_0, ω) is a tuple $M^{(\Gamma_0,\omega)} = \langle W', \sim', \leqslant', V' \rangle$ where $\sim'$ and V' are defined as in Definition 3.3(ii) and (iii), W' and $\leqslant'$ are defined as follows:

(i) $W' = W \cup \{(\Gamma_0, \omega)\}$, where $W = \{(\Delta, \psi) \mid \Delta \textit{ is a MCS and } \Gamma_0^{\psi} \subseteq \Delta\}$;

(ii) $\leqslant' = \leqslant \cup \{\langle(\Gamma_0, \omega), (\Gamma_0, \omega)\rangle\} \cup \{\langle(\Gamma_0, \omega), (\Delta, \psi)\rangle \mid (\Delta, \psi) \in W\}$, where $\leqslant$ is defined as in Definition 3.3(iii).

The truth lemma and verification lemma for Limiting case can be proved easily based on Lemma 3.5.

Theorem 3.8 $\mathbb{KCDL}$ *is strongly complete with respect to the class of epistemic betterness structures that are reflexive, transitive and $\sim$-smooth.*

We observe that it is straightforward to redeploy the above completeness argument to prove the strong completeness of $\mathbf{F}+(CM)$ with respect to the class of betterness structures that is reflexive, transitive, smooth and where $\sim$ is universal. Such completeness result was left as an open question in [6] and is also the focus of a forthcoming publication by Parent [2].

4 Conclusions

We define a new dyadic deontic operator to describe the knowledge-based conditional obligations and provide a sound and strongly complete logic for it with respect to the reflexive, transitive and $\sim$-smooth epistemic betterness structures.

References

[1] Åqvist, L., "Introduction to Deontic Logic and the Theory of Normative Systems," Bibliopolis, 1987.

[2] Gabbay, D., J. Horty, X. Parent, R. van der Meyden and L. van der Torre, *Handbook of deontic logic and normative systems* **2** (forthcoming).

[3] Hansson, B., *An analysis of some deontic logics*, Nous (1969), pp. 373–398.

[4] Kraus, S., D. Lehmann and M. Magidor, *Nonmonotonic reasoning, preferential models and cumulative logics*, Artificial intelligence **44** (1990), pp. 167–207.

[5] Pacuit, E., R. Parikh and E. Cogan, *The logic of knowledge based obligation*, Synthese **149** (2006), pp. 311–341.

[6] Parent, X., *Maximality vs. optimality in dyadic deontic logic*, Journal of Philosophical Logic **43** (2014), pp. 1101–1128.

[7] Prakken, H. and M. Sergot, *Dyadic deontic logic and contrary-to-duty obligations*, in: *Defeasible deontic logic*, Springer, 1997 pp. 223–262.

[8] Van Fraassen, B. C., *The logic of conditional obligation*, in: *Exact Philosophy*, Springer, 1973 pp. 151–172.

Another Proof of the Realization Theorem

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Abstract

The realization theorem connects modal logic with its explicit counterpart, justification logic, or logic of proofs, by relating occurrences of the modal operator in a modal theorem with suitable proof terms, and turning a modal theorem into a theorem in justification logic or logic of proofs. In this paper, we propose another proof of the realization theorem, focusing on the relation between S4 and LP. We will define a concept called *positive expansion* on modal formulas, and prove that through the expansion, every S4 theorem can be turned into a theorem whose realization is a theorem in LP^-, a subsystem of LP without +. Both semantic and syntactic proofs are given for this result, where the semantic proof also provides a structural analysis of the semantics of LP^-. Then an algorithmic procedure is provided which in a way reverses the procedure of expansion to convert a +-free realization of the expansion to a realization of the original S4 theorem in the system of LP.

Keywords: Modal logic, Justification Logic, Realization, Logic of Proofs.

1 Introduction

The realization theorem is a main result in the study of justification logic [2,1,8]. It provides a formal connection between assorted justification logical systems with their modal epistemic logical counterparts in a formal structural way. Granted the importance of the realization theorem, various proofs have been proposed. There is a constructive proof given in [2] concerning the first axiom system of justification logic, LP, treated as a logic of proofs, and its modal epistemic counterpart S4. The proof uses cut-free Gentzen style S4 proofs as a guide to establish the formal connection. The first semantic proof of the theorem is given in [6], which is also where the possible-world-like semantics for justification logic is introduced. The method used in the semantic proof is later extended by the author to suggest a two-stage proof procedure for an infinite class of justification logics [5]. More proofs of the theorem can be found in [4,3,7,9].

In this paper, we propose another proof of the realization theorem concerning the relation between S4 and LP. The importance and novelty of the proof rest on its revelation of the function of + in the procedure of realization. We will define a concept called *positive expansion* on modal formulas and prove that through the expansion, every S4 theorem can be turned into a theorem whose realization is a theorem in LP^-, the subsystem of LP without +. Then an algorithmic procedure is provided which in a way reverses the procedure of expansion to convert a +-free realization of the expansion to a realization of the initial S4 theorem in the system of LP. More clear statements of the result have to wait until some formal definitions are given. But roughly, the process of the expansion is to substitute formulas of the form $\Box X \vee \Box X$ for $\Box X$, which is the modal counterpart of the process of substituting the disjunction $s{:}\phi \vee t{:}\phi$ for $(s+t){:}\phi$. Thus the process of the expansion can be viewed as the process of removing + without + being explicitly stated, and this is justified by our proof that every S4 theorem can indeed be expanded to a modal theorem whose realization is +-free. Then adding + back to the realization of the expanded modal theorem, we can obtain a realization of the original analyzed modal theorem in the system of LP. For the proof of the realization of a positive expansion of an S4 theorem into an LP^- theorem, we provide both the semantic and syntactic proofs, where the semantic proof renders a structural analysis of the semantics of LP^-, and the syntactic proof gives us another view of how + functions in the procedure of realization.

2 Positive Expansion and the Realization Theorem

Some basic knowledge of justification logic and modal logic is assumed. The languages of S4, LP^-, and LP are denoted as $\mathcal{L}_{\mathsf{S4}}$, $\mathcal{L}_{\mathsf{LP}^-}$ and $\mathcal{L}_{\mathsf{LP}}$, respectively. Comparing the languages, we can see that a formula in $\mathcal{L}_{\mathsf{LP}}$, or $\mathcal{L}_{\mathsf{LP}^-}$ is in a way the result of filling in the occurrences of $\Box$ of an $\mathcal{L}_\Box$ formula with proof terms. We give a formal definition based on the observation.

Definition 2.1 Call a formula of the form $\Box G$ m-*formula*. Given a formula F in $\mathcal{L}_\Box$, $\mathcal{O}(F)$ denotes the set of occurrences of m-*formula* in F, and $\mathcal{O}^+(F)$ and $\mathcal{O}^-(F)$ the sets of positive occurrences and negative occurrences of m-formula in F respectively. So $\mathcal{O}(F) = \mathcal{O}^+(F) \cup \mathcal{O}^-(F)$. [1]

Definition 2.2 Given a formula $F \in \mathcal{L}_\Box$, a *proof term assignment*, *pt-assignment*, on F assigns a proof term to an occurrence in $\mathcal{O}(F)$.

Definition 2.3 Let $F \in \mathcal{L}_\Box$, and $\mathcal{R}(F)$ be the set of pt-assignments of F. Also let $* \in \{+, -\}$.

(i) $r \in \mathcal{R}^*_{\mathbf{n}}(F) \subseteq \mathcal{R}(F)$ if and only if $r(\mathcal{O}^*(F)) \subseteq \mathcal{V}$, where $\mathcal{V}$ is the set of propositional variables;

(ii) $r \in \mathcal{R}^*_{\mathbf{sn}}(F) \subseteq \mathcal{R}^*_{\mathbf{n}}(F)$ if and only if the restriction of r to $\mathcal{O}^*(F)$, $r|\mathcal{O}^*(F)$, is injective. where $r|\mathcal{O}^*(F)$ is the restriction of r to $\mathcal{O}^*(F)$.

A pt-assignment r is *positive normalized* if $r \in \mathcal{R}^+_{\mathbf{n}}(F)$, and *negative normalized* if $r \in \mathcal{R}^-_{\mathbf{n}}(F)$; then we call a pt-assignment r *strictly positive normalized* if $r \in \mathcal{R}^+_{\mathbf{sn}}(F)$, and *strictly negative normalized* if $r \in \mathcal{R}^-_{\mathbf{sn}}(F)$.

[1] Basically formula occurences and the sets of $\mathcal{O}(F)$, $\mathcal{O}^+(F)$ and $\mathcal{O}^-(F)$ can be formally defined. An example of the definition can be found in [9].

Definition 2.4 Given formulas $F, G \in \mathcal{L}_\Box$, by $F \prec_1 G$, we mean $G = F[\Box X/\ \Box X \vee \Box X]$, that is, G is the result of substituting $\Box X \vee \Box X$ for an occurrence of $\Box X$ in F. We write $F \prec_1^+ G$ and $F \prec_1^- G$ to indicate that the occurrence of $\Box X$ in F is positive and negative respectively. Furthermore, $\preccurlyeq$, $\preccurlyeq^+$, and $\preccurlyeq^-$ are the transitive and reflexive closure of $\prec_1$, $\prec_1^+$, and $\prec_1^-$, respectively, and we call $F \preccurlyeq^+ G$ that G is a positive expansion of F, and $F \preccurlyeq^- G$ negative expansion.

Definition 2.5 Given $F {\prec_1}^+ G$ ($F {\prec_1}^- G$ respectively) with $G = F[\Box X/\ \Box X \vee \Box X]$ for some positive (negative) occurrence $\Box X$ in F, and a function $\mathsf{p}\colon \mathcal{O}^-(F) \mapsto \mathcal{T}$ ($\mathsf{p}\colon \mathcal{O}^+(F) \mapsto \mathcal{T}$), we say a pt-assignment r on G is *rooted in* p if and only if there is a pt-assignment r' on F such that $r'|\mathcal{O}^-(F)$ ($r'|\mathcal{O}^+(F)$) is p, and $G^r = F^{r'}[(\Box X)^{r_1}/(\Box X)^{r_2} \vee (\Box X)^{r_3}]$, where all the restrictions of r_1, r_2, and r_3 to their respective $\mathcal{O}^-(\Box X)$ ($\mathcal{O}^+(\Box X)$) are equal to $\mathsf{p}|\mathcal{O}^-(\Box X)$ ($\mathsf{p}|\mathcal{O}^+(\Box X)$ respectively).

Here's an example. Let P and Q be propositional variables. If F is $\Box P \to \Box\neg\Box Q$ and G is $\Box P \to (\Box\neg\Box Q \vee \Box\neg\Box Q)$, then $F \prec_1^+ G$ with $\Box X = \Box\neg\Box Q$; and if $F^{r'}$ is $x{:}P \to t_1{:}\neg y{:}Q$, and G^r is $x{:}P \to (t_2{:}\neg y{:}Q \vee t_3{:}\neg y{:}Q)$, then r' is a strictly negative normalized pt-assignment on F provided x and y are distinct variables, and the pt-assignment r on G is rooted in $r'|\mathcal{O}^-(F)$. Notice that y is duplicated in G^r, and t_1, t_2 and t_3 are not necessary to be equal.

Definition 2.6 Given formulas $F, G \in \mathcal{L}_\Box$ and G being a positive (negative) expansion of F, a pt-assignment r on G is *rooted in* $\mathsf{p}\colon \mathcal{O}^-(F) \mapsto \mathcal{T}$ ($\mathsf{p}\colon \mathcal{O}^+(F) \mapsto \mathcal{T}$) if there is a sequence of formulas $F{=}F_0, \ldots, F_n{=}G$ and a sequence of pt-assignments $r_0 \in \mathcal{R}(F_0), \ldots, r_n \in \mathcal{R}(F_n)$ such that $r_0|\mathcal{O}^-(F)$ ($r_0|\mathcal{O}^+(F)$) is p, $F_{i-1} \prec_1^+ F_i$ ($F_{i-1} \prec_1^- F_i$), and r_i is rooted in $r_{i-1}|\mathcal{O}^-(F_{i-1})$ ($r_{i-1}|\mathcal{O}^+(F_{i-1})$), for $1 \le i \le n$.

Assume that the constant specifications are axiomatically appropriate and term-schematic. Given our notations, the realization theorem is as follows:

Theorem 2.7 *F is an* S4 *theorem if and only if there is a pt-assignment r on F such that F^r is an* LP *theorem.*

There are two directions in the theorem. The one from right to left is the easy one. We focus on the other, in which a stronger result that r is strictly positive normalized can be obtained. In this case, F^r is called a *normal realization* in the literature. Our proof is through the following two theorems:

Theorem 2.8 *If F is an* S4 *theorem, then there is a positive expansion G of F, and a pt-assignment r on G rooted in an injective $\mathsf{p}\colon \mathcal{O}^-(F) \mapsto \mathcal{V}$ such that G^r is* LP^- *provable.*

Theorem 2.9 *Given a positive expansion G of F, and a pt-assignment r on G rooted in an injective $\mathsf{p}\colon \mathcal{O}^-(F) \mapsto \mathcal{V}$, there is a substitution σ and a strictly positive pt-assignment r' on F such that $G^{r\sigma} \to F^{r'}$ is* LP *provable.*

Call a formula $F \in \mathcal{L}_\Box$ with a pt-assignment r such that F^r is LP^- provable a *strong theorem*. Then Theorem 2.8 tells us that F is an S4 theorem if and only if there is a formula G with $F \preccurlyeq^+ G$, such that G is a strong theorem; and furthermore, among all the pt-assignments r on G such that G^r is LP^- provable, we can pick out an r' which is rooted in an injective $\mathsf{p}\colon \mathcal{O}^-(F) \mapsto \mathcal{V}$ such that $G^{r'}$ is an LP^- theorem. Then according to Theorem 2.9, there is a substitution σ, and a strictly negative normalized pt-assignment r'' of F such that $G^{r\sigma} \to F^{r''}$ is LP provable. Since we are working on an $\mathsf{LP}(\mathcal{CS})$ axiom system with $\mathcal{CS}$ term schematic, $G^{r\sigma}$ is LP provable, and so is $F^{r''}$. This gives us the realization theorem.

3 Constructive Method

Now we analyze the first proof of the realization theorem given in [2] to provide constructive proof of Theorem 2.8. We call a Gentzen style $\mathsf{S4}$ proof *strong* if every family in the proof contains at most one essential occurrence. Note that following the original realization procedure in [2], the conclusion of a strong proof is realized to a plus-free normal realization. Suppose that a Gentzen style proof is not strong. We can pick out an essential family of the proof in which an occurrence of $\Box F$ in the conclusion of a rule is related to two occurrences of $\Box F$ in the premise(s) with each of them belonging to an essential family of the subproof tree(s) of the premise(s). There could be one or two subproof trees, depending on that it is a contraction rule or a two-premise rule. We call the essential family on the left, family 0, and on the right, family 1. Then turn the whole proof into a new one by substituting $\Box F \vee \Box F$ for all the occurrences of $\Box F$ in the essential family that we just pick out. In this procedure, certainly $Y \prec_1^+ Z$, if Z is the resulting formula of the substitution from the formula Y.

It can then be easily checked that every application of the rules is still an application of the same rule, except the applications of the right modal rule in which $\Box F$ is introduced. Then we have such an instance in the proof tree:

$$\frac{\Box\Gamma \Rightarrow F}{\Box\Gamma \Rightarrow \Box F \vee \Box F},$$

which is then replaced by the following:

$$\cfrac{\cfrac{\Box\Gamma \Rightarrow F}{\Box\Gamma \Rightarrow \Box F}\ \mathrm{R}\Box}{\Box\Gamma \Rightarrow \Box F \vee \Box F}\ \mathrm{R}\vee_i,$$

where i=0 or 1 depending on that $\Box F$ is in family 0 or 1. Continue the process, we will eventually have a strong proof, and the conclusion is a strong theorem expanded from the original $\mathsf{S4}$ theorem. Finally, since the proof now is strong, applying the original algorithm given in [2], we have a realization for the strong theorem.

4 Comparison

In [5], Fitting, extending from his previous work, proposed a universal method to deal with the realization problem for an infinite class of logics, including all the justification logic counterparts of Geach logics. Both Fitting's method and the realization procedure adopted here take two stages, with the first at which an LP^- theorem related in some way to an analyzed modal theorem ϕ

is produced: in [5], it's the quasi-realization of ϕ, and here, the realization of a positive expansion of ϕ, and with the second at which an algorithm is provided to turn the quasi-realization or the realization of the positive expansion to a realization of ϕ in LP. In our procedure, the LP^- theorem produced at the first stage is a realization of a modal theorem whose structure compared with the originally analyzed modal theorem ϕ is given beforehand: it is a positive expansion of ϕ; on the other hand, in Fitting's method, no such knowledge is provided; the existence of a quasi-realization of ϕ is justified by directly examining all possible combinations of the realizations of subformulas of the analyzed modal theorem. Technically, such knowledge of the comparative structure between the underlying modal theorems of the realizations simplifies the algorithm given at the second stage. Compared with the complication of the algorithm of turning a quasi-realization into a realization, the one given at the second stage in our procedure which is guided by the process of the expansion is relatively simple. Furthermore, only in such an algorithm in which $+$ is used in a way against the structures of the modal theorems, the function of $+$ is fully revealed. In a nutshell, $+$ is added to the realization of the positive expansion of an analyzed modal theorem to group together realizations of formulas which are duplicated in the process of the expansion. Such a function of $+$ can be clearly viewed by comparing the original proof of the realization theorem in [2] and the constructive proof given here, and this investigation of the function of $+$ can be generalized to concern the realization of the other justification logic by the semantic method given in this paper.

References

[1] Artemov, S. and M. Fitting, "Justification Logic: Reasoning with Reasons," Cambridge University Press, 2019.

[2] Artemov, S. N., *Explicit provability and constructive semantics*, Bulletin of Symbolic logic (2001), pp. 1–36.

[3] Fitting, M., *Realization implemented*, Technical report, Technical Report TR–2013005, CUNY Ph. D. Program in Computer Science (2013).

[4] Fitting, M., *Realization using the model existence theorem*, Journal of Logic and Computation (2013), p. ext025.

[5] Fitting, M., *Modal logics, justification logics, and realization*, Annals of Pure and Applied Logic **167** (2016), pp. 615–648.

[6] Fitting, M. C., *The logic of proofs, semantically*, Annals of Pure and Applied Logic **132** (2005), pp. 1–25.

[7] Goetschi, R. and R. Kuznets, *Realization for justification logics via nested sequents: Modularity through embedding*, Annals of Pure and Applied Logic **163** (2012), pp. 1271–1298.

[8] Kuznets, R. and T. Studer, "Logics of Proofs and Justifications," College Publications Cambridge, 2019.

[9] Wang, R.-J., *Non-circular proofs and proof realization in modal logic*, Annals of Pure and Applied Logic **165** (2014), pp. 1318–1338.